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Numerical Study of Inverse Problem with Computing the Functions and Parameters of Fractional Heat Equation Using the Transform of α-Fractional Derivative

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دراسة عددية للمسألة العكسية في حساب دوال ومعايير المعادلة الحرارية الكسرية باستخدام تحويل مشتقة ألفا الكسرية

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Abstract:
The inverse problem (IP) of the differential equations focusses for determining an unknown parameter(s) or a function. However, the IP of the fractional partial differential equations (FPDEs) problem plays a big role in engineering and applied science. Accordingly, the fractional differential equations (FDEs) have the significant rule in the mathematical modeling of science and engineering. As well as, finding the solutions of the FPDEs is a significant subject and a wide field. The objectives of this article is to study the method of solutions of the IP for determining unknown parameters or functions of the problem of FPDEs by using the definition of α-fractional derivative transform which is converted a FPDEs to a partial differential equation (PDE) and then, we can use the method of lines (MOL) with finite difference method for solving a quasi-linear PDE by converted it to an ordinary-differential equations (ODEs) system. The characteristic of α-fractional derivative transform is very appropriate, significant, and powerful for solving the problems of FPDEs. Additionally, the useful properties of the definition of α-fractional derivative transform are used in converting the quasi-linear FPDEs to a quasi-linear PDE. Hence, it is converted to a system of ODEs by using the MOL and finite difference formulas. Some implementations of inverse problems of the FPDEs are solved using the proposed method and then, they have compared with the numerical solutions. The test implementations showed that the two approximated solutions using the proposed method are identical. Hence, the algorithm of the proposed method proved to be efficient and accurate.

Keywords: Inverse Problem, α-Fractional Derivative; System of ODEs; IP; PDEs; FPDEs,

الملخص
تركز المسألة العكسية (IP) للمعادلات التفاضلية على تحديد المجهول المعلمة (المعلمات) أو الدالة ومع ذلك، تلعب IP من المعادلات التفاضلية الجزئية الكسرية (FPDEs) دورًا كبيرًا في الهندسة والعلوم التطبيقية. وبناء على ذلك فإن المعادلات التفاضلية الكسرية (FDEs) تتمتع بقاعدة مهمة في النمذجة الرياضية للعلوم والهندسة. بالإضافة إلى ذلك، يعد إيجاد حلول لـ FPDEs أمرًا مهمًا موضوع ومجال واسع. ان أهداف هذه المقالة هي دراسة طريقة حلول IP لتحديد المعلمات أو الوظائف غير المعروفة لمشكلة FPDEs باستخدام تعريف تحويل المشتق الكسري α والذي يتم تحويل FPDEs إلى جزئي المعادلة التفاضلية (PDE) ومن ثم يمكننا استخدام طريقة الخطوط (MOL) ذات النهاية المحدودة طريقة الفرق لحل PDE شبه الخطي عن طريق تحويله إلى نظام معادلات تفاضلية اعتيادية (ODEs). إن خاصية تحويل المشتق الكسري α مميزة للغاية ومناسبة وهامة وقوية لحل مشاكل FPDEs. بالإضافة إلى ذلك، يتم استخدام الخصائص المفيدة لتعريف تحويل المشتق الكسري α في تحويل FPDEs شبه الخطية إلى PDE شبه الخطية. ومن ثم يتم

تحويله إلى نظام ODEs بواسطة باستخدام صيغ MOL والفرق المحدود. بعض تطبيقات المشاكل العكسية تم حل FPDEs باستخدام الطريقة المقترحة ومن ثم مقارنتها مع الحلول العددية. وأظهرت تطبيقات الاختبار أن الحلين التقريبيين باستخدام الطريقة المقترحة متطابقة. ومن هنا أثبتت خوارزمية الطريقة المقترحة لتكون فعالة ودقيقة.

الكلمات المفتاحية: المسألة العكسية، مشتقة كسورية α؛ نظام ODEs. IP؛ PDEs ؛ FPDE.

Introduction

In the fields of engineering and applied sciences such as mathematics, physics, control, mathematical modeling, optimization, chemical engineering, and medical engineering, an inverse problem (IP) of differential equations is an essential hot problem where the objective of the topic of this research of this topic is to determine the source as parameters or functions which are underlying system that originated a specific collection of observations or consequences. The IP involves of the determining the values of the parameters or the functions which describing the system of the study from the actual results of specific cause evaluations. Furthermore, because the features of the sources in real problems are never set and are necessarily constantly invention in this particular case additionally depends on measurement-based indirectly observable information. In contrast, IP starts with a known cause and asks the researcher to determine what effects it will have. To ascertain the conditions of the research difficulty, some academics have studied inverse problems since exactly fifty years ago. However, inverse challenges of determining source terms since the 1970s are extensively researched since sources attributes are rarely identified in practical applications whereas the heat equation's inverse problem is therefore recognizable as an inverse control problem, wherein the goal is to identify which parameter or function(s) of source control, at any given time, results in the desired temperature at a particular position x_0 in the spatial domain. Although the problem's solvability is fairly established.

Furthermore, in the 20th and 21th centuries, fractional calculus received a lot of interest, especially within the last three years. It is frequently employed in the fields of engineering and applied science, as well as in applied and pure mathematics. Numerous authors have looked into some FDEs new concepts and definitions. For example, they have developed the theory and the applications of FDEs in addition to the methods of solutions for solving FDEs and specified some novel concepts in the definitions of fractional derivatives (FDs) [1-5].

Nonetheless, a number of researchers provided some fresh interpretations of FDs. In light of this, the literature overview on FDs and FDE solutions can be introduced for example as follows: Khalil et al. [6] presented a novel FD's definition, and Mechee et al. [7] provided an important definition of the α-fractional integral and the α-FD of real functions. Furthermore, Zheng et al. defined the Caputo type for FD and investigated its characteristics [8]. In addition to, the conformable fractional differential transform (CFDT) and its application were initially introduced by Unal and Gan [9]. Nevertheless, the second-order conjugate boundary value problems (BVPs) have been modified by Anderson and Avery using the new definition of FD [10], and Khalil and Hammad have looked at Legendre conformable FDEs and their basic features [11]. Likewise, Abdel Hakim [12] verified the existence of the conformable fractional derivative. Khalil and Abu-Hammad investigated precisely the response to the heat-conformable FDE. Additionally, Abdel Jawad proposed the notion of the conformable FD after establishing the basic ideas of FDs [13]. Furthermore, the conformable FD features were offered to Ortega and Rosales [14], Similar to the IPs, the IP for recognizing an unknown heat-source parameter or function in the heat conduction equation has been the topic of numerous investigations [15–22]. As a result, Cannon, Duchateau, and Fatullayev investigated the inverse source problem with additional data for f = f(u) [15, 16]. In a similar fashion, the problems of determining the source function in a parabolic equation and nonhomogeneous heat equation have been introduced by [17-18]. As a consequence, several numerical techniques for solving the inverse source problem, have been presented [19–26].

The current study proposes that the heat source only depends on time, and the overdetermination can be credited to the transient temperature measurement obtained from a single thermocouple that is positioned within the heat conductor. For this purpose, we will solve the inverse problem which include FPDE using some of the characteristics of α-fractional derivative transform which converted to PDE. Furthermore, we apply the MOL technique for solve PDE in this inverted source problem in this paper. Hence, MOL with finite difference method have been companied for solving a quasi-linear PDE by converted it to a system of ODEs. Therefore, a quasi-linear PDE was successfully solved through the use of MOL and the finite differences approach, which converts it into a system of ODEs.

Background of Fractional Derivatives

Some concepts and definitions relevant to this subject have been introduced in this section. The classical and modern definitions of FDs of the function f(τ) have introduced.

Firstly, the classical of Riemann-Liouville fractional derivative of the function f(τ) [1] is defined in the following definition.

Definition 1: Riemann-Liouville Derivative

Riemann-Liouville Derivative [1] defined the classical fractional Riemann-Liouville integral operator of left side with order α > 0 as follows:

$$ {}^{a}D_{\tau}^{\alpha}\, \emptyset(\gamma) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^{\gamma} \frac{\emptyset^{(m)}(\gamma)}{(\gamma-\mu)^{m-\alpha-1}}\, d\mu, \quad (1) $$

where the fractional number α satisfy the inequality $m - 1 \leq \alpha < m$ where $m \in N$ and the function $\emptyset: [a,\infty) \to \mathcal{R}$ is continuous in its domain.

Secondly, two modern fractional derivatives of the function f(τ) [6-7] are defined in the following two definitions:

Definition 2: Comfortable Fractional Drivative

Khalil et. al [6] introduced the comfortable FD of the function $\emptyset(\gamma) : [a,\infty) \to \mathcal{R}$. as follows:

$$ T_{\alpha}\big(\emptyset(\gamma)\big) = \emptyset^{(\alpha)}(\gamma) = \lim_{\epsilon \to 0} \frac{\emptyset(\gamma + \epsilon\gamma^{\,1-\alpha}) - \emptyset(\gamma)}{2\epsilon}. \quad (2) $$

for $\alpha \in (0,1]$.

Definition 3: α-Fractional Drivative

Mechee et al. [7] has introduced α-FD of the function $f(\tau)$ is as follows:

$$ T_{\alpha}\big(\emptyset(\gamma)\big) = \emptyset^{(\alpha)}(\gamma) = \lim_{\epsilon \to 0} \frac{\emptyset(\gamma + \epsilon\gamma^{\,1-\alpha}) - \emptyset(\gamma - \epsilon\gamma^{\,1-\alpha})}{2\epsilon}. \quad (3) $$

Theorem 1 [6]

If $g : [a,\infty) \to \mathcal{R}$ is a real function in the domain I_{α} then, the following property of α-fractional derivative of this function g is obtained from the Definitions 2 and 3 of α-fractional derivative for the function $g(\tau)$:

$$ T_{\alpha}\big(g(\tau)\big) = g^{(\alpha)}(\tau) = \tau^{1-\alpha}\, g'(\tau). \quad (4) $$

Proposition 1: Let $u(\tau, x)$ be differential function in the domain $\mathfrak{H} \sqsubset \mathbb{R} \times \mathbb{R}$

Then, $\frac{\partial^{\alpha} u(\tau,x)}{\partial \tau^{\alpha}} = \tau^{1-\alpha}\, u_{\tau}(\tau, x)$ and $\frac{\partial^{\alpha} u(\tau,x)}{\partial x^{\alpha}} = x^{1-\alpha}\, u_{x}(\tau, x)$.

Proof

From the definitions in the Equation (2) and (3), we get the following

$$ T_{\alpha}\big(\emptyset(\tau)\big) = \emptyset^{(\alpha)}(\tau) = \tau^{\,1-\alpha}\ \emptyset'(\tau). $$

However, we can convert the fractional partial derivative to a partial derivative as follows:

$$ \frac{\partial^{\alpha} u(\tau, x)}{\partial \tau^{\alpha}} = \lim_{\epsilon \to 0} \frac{u(\tau + \epsilon\tau^{\,1-\alpha}, x) - u(\tau, x)}{\epsilon}, $$

$$ = \lim_{\epsilon \to 0} \frac{u(\tau, x) + \epsilon\tau^{\,1-\alpha} u_{\tau}(\tau, x) + \frac{(\epsilon\tau^{\,1-\alpha})^2}{2!} u_{\tau\tau}(\tau, x) + \cdots - u(\tau, x)}{\epsilon}. $$

Then, $\frac{\partial^{\alpha} u(\tau,x)}{\partial \tau^{\alpha}} = \tau^{1-\alpha}\, u_{\tau}(\tau, x)$.

In the same way, we can prove the second relation

$$ \frac{\partial^{\alpha} u(\tau, x)}{\partial x^{\alpha}} = x^{1-\alpha}\, u_{x}(\tau, x). \quad (5) $$

Fractional Diffusion Equation

In this section, fractional diffusion equations have been introduced

Fractional Quasi-Linear FPDE

The special quasi-linear second-order FPDE has the following form

$$\frac{\partial^\alpha w(\mathrm{t},\xi)}{\partial \mathrm{t}^\alpha} = \emptyset\left(w(\mathrm{t},\xi), \frac{\partial w(\mathrm{t},\xi)}{\partial \xi}, \frac{\partial^2 w(\mathrm{t},\xi)}{\partial \xi^2}\right), \quad 0 < \xi < l, \mathrm{t} > 0, \tag{6}$$

with the initial-condition (IC)

$$\mathrm{w}(0,\xi) = \mathrm{f}(\xi), \text{ for } 0 < \xi < l, \tag{7}$$

and the Dirichlet-boundary conditions (DBCs)

$$\mathrm{w}(\mathrm{t},0) = \mathrm{w}(\tau,\mathrm{l}) = 0 \text{ for } \mathrm{t} > 0. \tag{8}$$

Fractional Non-Homogenous Heat Equation

In special case consider a nonhomogeneous variable-coefficient fractional heat equation in one dimension:

$$\frac{\partial^\alpha w(\tau,\xi)}{\partial \tau^\alpha} = \beta^2\, \tau^{\,\alpha-1} \frac{\partial^2 w(\tau,\xi)}{\partial \xi^2} + f(\tau), \qquad 0 < \xi < l, \qquad \tau > 0, \tag{9}$$

with the IC and DBCs in Equations (7) and (8)

The FPDE in Equation (9) is non homogenous fractional heat equation with variable coefficient.

Proposed Method of Converting the FPDEs to PDEs Using α-Fractional Derivative Transform

From the proposition 1, the FPDE in Equation (6) is converted to the FPDEs

$$\mathrm{w}_\tau(\tau,x) = \tau^{\,\alpha-1}\, \emptyset\, \left(w(\tau,\xi), \frac{\partial u(\tau,\xi)}{\partial \xi}, \frac{\partial^2 u(\tau,\xi)}{\partial \xi^2}\right), \quad 0 < x < l, \tau > 0, \tag{10}$$

In the same way Equation (9) is converted to

$$\mathrm{w}_\tau(\tau,x) = \beta^2\, \frac{\partial^2 w(\tau,x)}{\partial x^2} + F(\tau), \tag{11}$$

where $F(\tau) = \tau^{\,\alpha-1} f(\tau)$. Equation (11) is transformed into the following homogenous heat equation by the assumption $v(\tau,\mathrm{x}) = w(\tau,\mathrm{x}) - r(\tau)$ where $r(\tau) = \int F(\tau)\, d\tau$, we get the following heat equation

$$v_\tau(\tau,x) = \beta^2\, \frac{\partial^2 v(t,x)}{\partial x^2}, \qquad (\tau,\mathrm{x}) \in (0,T] \times (0,1), \tag{12}$$

with the IC

$$v(0,\xi) = \mathrm{f}(\xi) - r(0) \text{ for } 0 < \xi < l, \tag{13}$$

and the DBCs

$$\mathrm{v}(\tau,0) = \mathrm{v}(\tau,\mathrm{l}) = r(\tau) \text{ for } \tau > 0. \tag{14}$$

Formulation of Inverse Problem of FPDEs

The following heat equation explains the diffusion, motion, and conduction of natural materials:

$$w_{\tau}(x,\tau) = \beta^2\, \Delta^2 w(x,\tau) + f(x,\tau,w), \qquad (x,\tau) \in \Omega \times (0,T],$$

subject to the IC

$$w(x,0) = w_0(x),\ 0 \le x \le 1$$

and the BCs

$$w(0,\tau) = g_0(\tau),$$
$$w(1,\tau) = g_1(\tau), \qquad 0 < \tau \le T$$

and the over specified condition

$$w(x_0,\tau) = h(\tau).\ 0 < \tau \le T$$

where $x_0 \in (0,1)$ indicates a thermocouple's internal position that records temperature and the functions $w_0(x)$, $g_0(\tau)$, $g_1(\tau)$ *and* $h(\tau)$ are given and satisfying the following compatibility conditions $w_0(\tau) = g_0(\tau)$, $w_n(\tau) = g_1(\tau), w(x_0,\tau) = h(\tau)$, and $w(x,\tau)$ with $f(\tau)$ are unknown functions, where the state variable is represented by $w(x,\tau)$ and β = the diffusion coefficient. Also, Ω is a bounded domain in R^d, and f refers to physical laws, or phrases that come from sources.

In this paper, we examine a one-dimensional time-dependent problem where the source is $f(x,\tau;\ u) = f(\tau)$ depends on time only for the quasi-linear FPDEs in Equation (9) with the IC and BCs in Equations (7)-(8). In special case consider a fractional-order heat equation in one dimension Equation (9) is transformed into the Equation (12) with the IC and BCs in Equations (13)-(14) and the over specification at a point $x_0 : v(x_0,\tau) = h(\tau) - r(\tau),\ 0 \le \tau \le T$ where the functions $w(x,\tau)$ and $f(\tau)$ are unknown functions satisfying the compatibility conditions.

Proposed Method

For solving the quasi-linear FPDE in Equation (9) with the IC and DBCs in Equations (7) and (8). The proposed method is converted this equation to a PDE in Equation (12) with the IC and DBCs in Equations (13)-(14) and then, this PDE solved by combining the MOL with the finite difference method. The following steps of the algorithm should be do.

Proposed Algorithm

The algorithm of the proposed method has the following steps:

1. Convert the FPDE in Equation (9) with its ICs and BCs in Equations (7) and (8) respectively to the heat equation in Equation (12) with the IC and DBCs in Equations (13)-(14) respectively by α-fractional derivative transform.
2. Use the MOL method with finite difference method to convert to Heat equation in Equation (12) to system of ODEs with the IC and BCs in Equation (13) and (14), and then, use a numerical method for solving this system as in steps 3-7.
3. Divide the domain of the problem in the variable x in [0, l] by n subinterval with the norm of partition $h = \frac{l}{n}$ and in the variable τ in [0, T] by m subinterval with the norm of partition $k = \frac{T}{m}$.
4. Do steps 5-7 while $1 \le i \le m$.
5. Put instead the point $v(\tau,x)$ by $v_{ij} = (\tau_i, x_j)$ of the PDE in Equation (12) for i=1, 2, …, m and j=0,1, 2, …, n.
6. Fix $\tau = \tau_i$ at the left side of Equation (12) and put the formulas of finite difference in the derivatives in the right side which converting the PDE in Equation (12) to following system of ODEs:
   $$v_i'(\tau_j) = \tau_j^{\ \alpha-1}\, \emptyset(v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}), \tag{15}$$
   with the following IC, and DBCs:
   $$v_{0j} = v(0,x_j) = f(x_j) - r(0)\ ;$$
   and
   $$v_{j0} = v(\tau_j,0) = v_{jn} = v(\tau_j,1) = r(\tau_j),\ \ 1 \le j \le m, \tag{16}$$
   respectively.
7. Solve the system of ODEs of first-order in Equation (15) with IC and B. Cs in Equation (16) using the RK-type method.

In general, this algorithm can generalize for solving a FPDE of n^{th}-order with I.Cs. and B.Cs.

Implementation:

Some examples which used to prove the efficient of the proposed method have been implemented in this section.

Numerical Examples

Using the proposed approach for three examples of testing, we provide and discuss numerical results in this section. T=1 has been used in these examples. The current approach is extremely successful according to the results.

Example 1: Consider the following inverse problem of FPDE of α-order where $\alpha = \frac{1}{2}$.

$$\frac{\partial^{\frac{1}{2}} u(\tau, x)}{\partial \tau^{\frac{1}{2}}} = \beta^2 \sqrt{\tau}\, \frac{\partial^2 u(\tau, x)}{\partial x^2} + \mathrm{f}(\tau), \qquad 0 < x < l, \;\; \tau > 0, \tag{17}$$

With the input data

IC $\quad \mathrm{u}(\mathrm{x}, 0) = u_0(\mathrm{x}) = \sin(\pi\, \mathrm{x}), \qquad 0 < x < l.$

BCs $\quad \mathrm{u}(0, \mathrm{t}) = g_0(\tau) = \mathrm{u}(1, \tau) = g_1(\tau) = \tau^2, \qquad 0 < \tau \le T.$

and the over specified condition: $\mathrm{u}(x_0, \tau) = \mathrm{h}(\tau) = e^{-\pi^2 t} \sin(\pi\, x_0) + \tau^2, \quad 0 < \tau \le T,$

and $\beta = 1$, where the functions $u_0(\mathrm{x})$, $g_0(\tau)$, $g_1(\tau)$ and $\mathrm{h}(\tau)$ are known functions but $\mathrm{u}(\mathrm{x}, \tau)$ and by $\mathrm{f}(\tau)$ are unknown functions. The inverse problem (17) has the unique solution given by f(τ) = 2τ.

Example 2: Consider the following FPDE of α-order where $\alpha = \frac{1}{4}$.

$$\frac{\partial^{\frac{1}{4}} u(\tau, x)}{\partial \tau^{\frac{1}{4}}} = \beta^2 \sqrt[4]{\tau^3}\, \frac{\partial u(\tau, x)}{\partial x} + 2\tau^{\frac{7}{4}}, \qquad 0 < x < l, \;\; \tau > 0, \tag{18}$$

IC $\quad \mathrm{u}(\mathrm{x}, 0) = u_0(\mathrm{x}) = 1 - e^{x}, \quad 0 < x < l.$

BCs $\quad \mathrm{u}(0, \tau) = g_0(\tau) = \tau^2 + 1 - e^{-\tau}, \mathrm{u}(1, \tau) = g_1(\tau) = \tau^2 + 1 - e^{1-\tau}, 0 < \tau \le T.$

and the over specified condition $\mathrm{u}(x_0, \tau) = \mathrm{h}(\tau) = \tau^2 + 1 - e^{x_0 - \tau}, \;\; 0 < \tau \le T \; and \; \beta = 1,$

where the functions $u_0(\mathrm{x})$, $g_0(\tau)$, $g_1(\tau)$ and $\mathrm{h}(\tau)$ are known functions but $\mathrm{u}(\mathrm{x}, \tau)$ and $\mathrm{f}(\tau)$ are unknown functions. The inverse problem (18) has the unique solution given by f(τ) = 2τ.

Example 3: Consider the following FPDE of α-order where $\alpha = \frac{3}{4}$.

$$\frac{\partial^{\frac{3}{4}} u(\tau, x)}{\partial \tau^{\frac{3}{4}}} = \beta^2\, {}^{4}\sqrt{\tau}\, \frac{\partial u(\tau, x)}{\partial x}, \qquad 0 < x < l, \;\; \tau > 0, \tag{19}$$

With the input data $\beta = 1$

IC $\quad \mathrm{u}(\mathrm{x}, 0) = u_0(\mathrm{x}) = e^{-2\mathrm{x}}, \qquad 0 < x < l.$

BCs $\quad \mathrm{u}(0, \tau) = g_0(\tau) = e^{2\tau}, \; \mathrm{u}(1, \tau) = g_1(\tau) = e^{2(1-\mathrm{x})}, 0 < \tau \le T.$

and the over specified condition $\mathrm{u}(x_0, \tau) = \mathrm{h}(\tau) = e^{2(\tau - x_0)}, \qquad 0 < \tau \le T.$

and $\beta = 1$, where the functions $u_0(\mathrm{x})$, $g_0(\tau)$, $g_1(\tau)$ and $\mathrm{h}(\tau)$ are known functions but $\mathrm{u}(\mathrm{x}, \tau)$ and $\mathrm{f}(\tau)$ are unknown functions. The inverse problem (19) has the unique solution given by f(τ) = 0.

Discussion and Conclusion

The proposed method converts the inverse problem of FPDE to an inverse problem of PDE by using α-fractional derivative transform and then, this PDF converts to a system of ODEs by combining MOL with finite difference formulas. This system of ODEs can be solved analytically or numerically. The numerical approach is used for solving this system of ODEs is the classical numerical RK method. This approach is used for solving three test problems, showing that they are agree well with the solutions. These implementations show the accuracy and efficiency of this approach. Accordingly, the approximated solutions in the Examples 1, 2 and 3 are calculated by using the proposed method by compound the MOL method with finite difference formulas and, the numerical RK methods of fourth and fifth-order. Meanwhile, the numerical solutions of these three examples are compared and plotted in Figure 1 for different three cases of τ for each example. Consequently, from the numerical comparisons for the numerical solutions of the FPDEs for Examples 1-3 in Figure 1, we can conclude the powerful and efficiency of the proposed α-FD method.

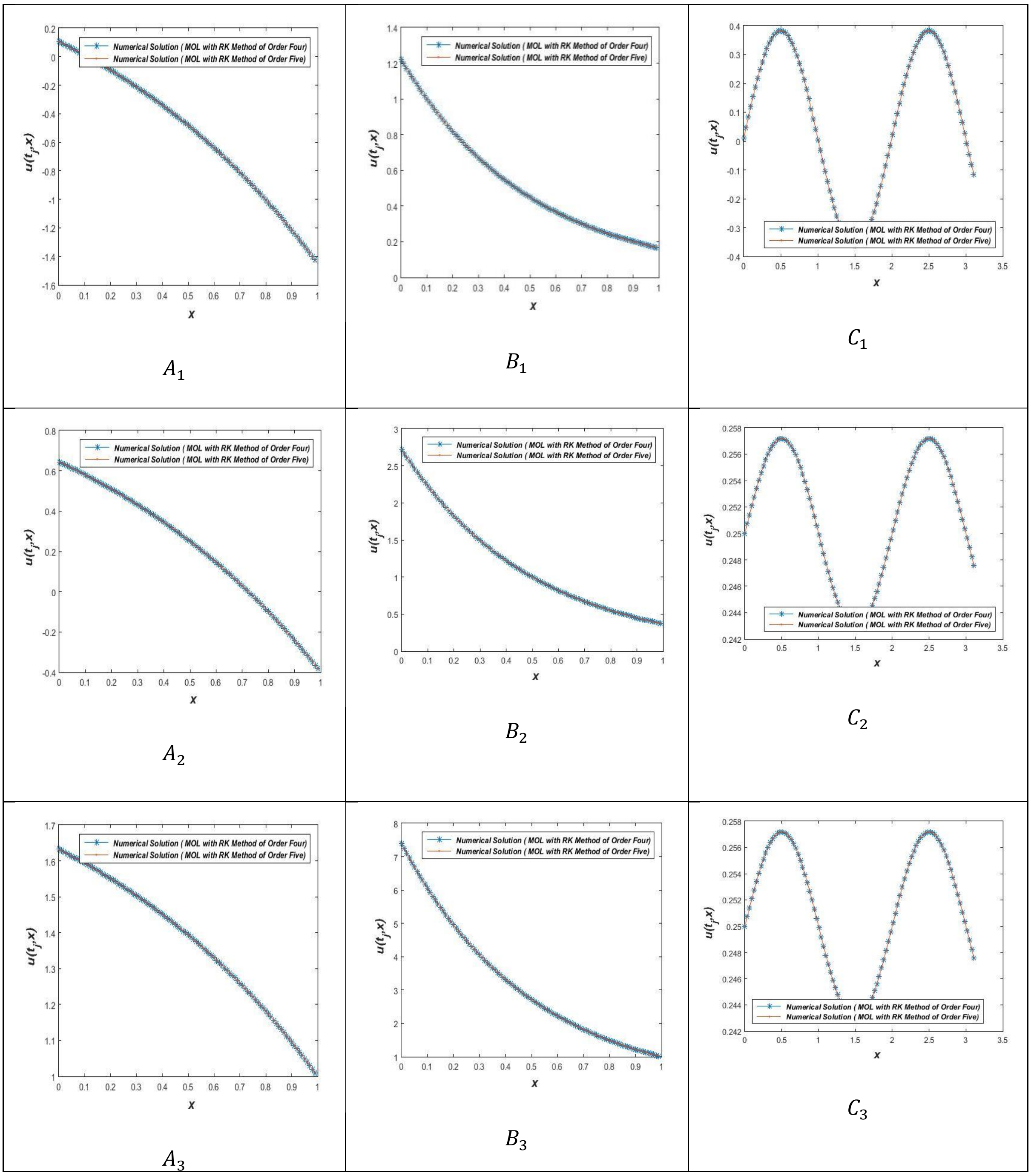

Figure 1: A Numerical Comparison of the Solutions of Example 1:((A1) $\tau = 0.1$, (A2) $\tau = 0.5$ and (A3) $\tau = 0.9$), Example 2:((B1) $\tau = 0.1$, B2) $\tau = 0.5$ and (B3) $\tau = 0.9$), and Example 3: ((C1) $\tau = 0.1$, (C2) $\tau = 0.5$ and (C3) $\tau = 0.9$) Using MOL with RK Methods of Fourth- and Fifth-order.

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