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w-Functions in Topological Spaces

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دوال من النمط w في الفضاءات التبولوجية

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Abstract:
The purpose of this research is to explore the properties of w-open sets and extend the concepts of interior and closure operators through these sets. Furthermore, we define new classes of functions in topological spaces, such as w-continuous, w-irresolute, w-open, and w-closed functions, and analyzing their characteristics and the relationships between them.

Keywords: w-open (closed) set, α-continuous function, semi-continuous function, pre-continuous function, κ-irresolute functions, for κ∈{α,semi,pre}.

الملخص
الهدف من هذا البحث هو تقديم ودراسة بعض الخواص للمجموعات المفتوحة من النمط w واستخدام هذا النمط من المجموعات لتعميم مفهوم مؤثر الداخلية ومؤثر الغلاقة. ونعرف صنف جديد من الدوال، تسمى الدوال المستمرة من النمط w والدوال الغير حاسمة من النمط w والدوال المفتوحة من النمط w والدوال المغلقة من النمط w في الفضاءات التبولوجية مع بعض الخواص والعلاقات التي تربط بينهم.

الكلمات المفتاحية: المجموعة المفتوحة(المغلقة) من النمط w، الدوال المستمرة من النمط α، الدوال المستمرة من النمط semi، الدوال المستمرة من النمط pre، الدوال غير الحاسمة من النمط κ حيث ∈{α,semi,pre} κ.

Introduction

The concept of open sets has undergone significant developments since the 1960s. Levine [1] extended this concept in 1963 by introducing semi-open sets. Later, Njastad [2] proposed a new class of generalized open sets, termed α-open sets, which lie between open and semi-open sets. During the early 1980s, Mashhour [3] studied preopen sets, while Abd El-Monsef [4] introduced β-open sets. In 1996, Andrijević [5] explored the idea of b-open sets.

In 2003, Pugh [6] introduced the definition of somewhere dense sets, while in 2022, K. Mira and M. Tarjam [7] presented w-open sets, accompanied by some of their properties.

The generalized open sets are often defined using interior and closure operators, and their study has attracted growing interest in the field of topology. Functions, particularly continuous ones, play a central role in mathematics. Over the years, various generalizations of continuous functions have emerged, such as α-continuous functions [12, 17], semi-continuity [1], pre-continuity [3], irresolute functions [11], α-irresoluteness [18], and pre-irresoluteness [15], κ-open (κ-closed [12,13,16]) functions [12,14,3], for $\kappa \in \{\alpha, \text{ sem}, \text{ pre}\}$.

This research aims to explore new types of functions, including w-continuous, w-irresolute, w-open, and w-closed functions, while examining the relationships between them.

Throughout this paper, for $A \subseteq X$, we denote $cl\,(A)$, $Int\,(A)$ for the closure, interior operator of A in X, respectively.

Definition 1.1. If $A \subseteq X$, then A is called:

1) α-open [2] if $A \subseteq int(cl(int(A)))$.
2) semi-open [1] if $A \subseteq cl(int(A))$.
3) pre-open [3] if $A \subseteq int(cl(A))$.
4) b-open [5] if $A \subseteq int(cl(A)) \cup cl(int(A))$.
5) β-open [4] if $A \subseteq cl(int(cl(A)))$.
6) somewhere dense [6] if $int(cl(A)) \neq \emptyset$.

Definition 1.2[7]. A subset $A \subseteq X$ is called w-open if there is a nonempty open set U such that $U \subseteq A$, i.e. $\mathrm{cl}(\mathrm{int}(\mathrm{A})) \neq \emptyset$. The complement of w-open is called w-closed.

Notation1.3. We write $\kappa(\tau)$ for the class of κ-open sets where $\kappa \in \{\alpha, \mathrm{semi}, \mathrm{pre}, \beta, \mathrm{b}, \mathrm{s}, \mathrm{w}\}$, s and w refer to somewhere dense and w-open, respectively.

Corollary 1.4[7]. In (X, τ), the following results hold:

1) A nonempty open (closed) set is w-open.
2) A nonempty semi-open set is w-open.
3) An w-open set is somewhere dense.

The following example clarifies that there is no relation among $\mathrm{pre}(\tau)$ and $w(\tau)$.

Example 1.5. Let $X = \{1, 2,3\}$ with $\tau_1 = \{X, \emptyset, \{1, 2\}\}$ and $\tau_2 = \{X, \emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Take $A = \{1, 3\}$, so $A \in \mathrm{pre}(\tau_1)$ but $A \notin \mathrm{w}(\tau_1)$, whereas $A \in \mathrm{w}(\tau_2)$ but $\mathrm{A} \notin \mathrm{pre}(\tau_2)$.

Theorem 1.6[7]. $A \subseteq X$ is w-closed $\Leftrightarrow A \subseteq F$ for some a proper closed subset F.

Notation 1.7. Generally, the intersection on $\mathrm{w}(\tau)$ is not closed, and the next example explains this.

Example 1.8. let $X = \{1, 2,3\}$ with $\tau = \{X, \emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and let $A = \{1,3\}, B = \{2,3\}$, so A, B $\in \mathrm{w}(\tau)$ whereas $A \cap B = \{3\} \notin \mathrm{w}(\tau)$.

The collection $\mathrm{w}(\tau)$ is closed under the union and the next theorem explains this.

Theorem 1.9. $\mathrm{w}(\tau)$ forms a supra on X.

Proof. If $A_i \in \mathrm{w}(\tau)$, then there exists $u_i \in \tau$ such that $u_i \subseteq A_i$, so $\cup\, u_i \subseteq \cup\, A_i$ and $\cup\, A_i \in w(\tau)$.

Definition 1.10[9, 10]. A space X is ultra-connected if the intersection of any two nonempty closed sets is nonempty. Equivalently, X is ultra-connected if the closures of distinct points always intersect.

Definition 1.11[9]. A space X is hyper-connected if the intersection of any two nonempty open sets is nonempty. Equivalently, X is hyper-connected if the closure of any open set is the entire space.

Definition 1.12[10]. A space X is F-connected if it is both hyper-connected and ultra-connected. So for any F-connected space X, there is an open subset of X which is contained in all others open sets.

Theorem 1.13[7]. If X is F- connected, then $w(\tau)$ forms a topology on X.

The following diagram shows the relationships between some of famous generalized open set.

$$\begin{tikzcd}
\text{non-empty open} \arrow[r] & \text{non-empty } \alpha\text{-open} \arrow[r] \arrow[d] & \text{non-empty semi-open} \arrow[rr] \arrow[d] & & \text{w-open} \arrow[d] \\
& \text{non-empty pre-open} \arrow[r] & \text{non-empty b-open} \arrow[r] & \text{non-empty } \beta\text{-open} \arrow[r] & \text{somewhere dense}
\end{tikzcd}$$

Definition 1.14. For $A \subseteq X$, then:

1) The w- interior of A (for short, $int_w(A)$) is the largest w-open set which is contained in A.
2) The w-closure of A (for short, $cl_w(A)$) is the smallest w-closed set which containing A.

Corollary 1.15. For A $\subseteq X$, the following results hold

1) $int(A) \subseteq int_w(A)$
2) $A \subseteq cl_w(A) \subseteq cl(A)$.

Proof. It is clear, using corollary 1.4(1)

Proposition 1.16. Suppose A and B are subset of (X, τ). Then:

1) $A \in w(\tau) \Leftrightarrow int_w(A) = A$.
2) A is w-closed $\Leftrightarrow cl_w(A) = A$.

3) If $A \subseteq B$, then $int_w(A) \subseteq int_w(B)$ and $cl_w(A) \subseteq cl_w(B)$.
4) $int_w(A) \cup int_w(B) \subseteq int_w(A \cup B)$ and $int_w(A \cap B) \subseteq int_w(A) \cap int_w(B)$.
5) $cl_w(A \cap B) \subseteq cl_w(A) \cap cl_w(B)$ and $cl_w(A) \cup cl_w(B) \subseteq cl_w(A \cup B)$.

Proof. It is clear.

The equality in the parts (4) and (5) do not hold in general, and the following examples explain this.

Example 1.17. Let $X = \mathbb{R}$ and let τ consists of all subsets of $\mathbb{R}$ that do not contain 0 or they have finite complement. Let $A = \{1,2,3,4\}, B = \{0\}$, $C = \mathbb{R}\backslash\{0\}$, $D = \{0, 1, 2\}$.

1) $int_w(A) = A$, $int_w(B) = \emptyset$, $int_w(A \cup B) = A \cup B$.
2) Since $cl(C) = \mathbb{R}$, so $cl_w(C) = \mathbb{R}$, then $cl_w(D) = D$ and $cl_w(C \cap D) = C \cap D$.

Example 1.18. If $X = \{1, 2, 3, 4\}$ with $\tau = \{X, \emptyset, \{1\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}\}$, and let $A = \{1, 4\}$, $B = \{3, 4\}$, $C = \{1\}$ and $D = \{3\}$, then , A, B, C and $D^c \in w(\tau)$ and so

1) $int_w(A) = A$, $int_w(B) = B$ and $int_w(A \cap B) = \emptyset$.
2) $cl_w(C) = C$, $cl_w(D) = D$ and $cl_w(C \cup D) = X$.

Theorem 1.19. Every superset of w-open is also an w-open set.

Proof. For $A \in \mathrm{w}(\tau)$, and $A \subseteq B$, there is an open set U in X where $U \subseteq A \subseteq B$, so $B \in \mathrm{w}(\tau)$.

Theorem 1.20. For (X, τ), we have:

$$\mathrm{w}(\tau) = \{U \cup A: \quad U \in \tau, \quad A \subseteq X\}$$

Proof. Since $U \cup int(A) \subseteq U \cup A$, so $U \cup A \in \mathrm{w}(\tau)$. Let $B \in \mathrm{w}(\tau)$, so there exists $U \in \tau$ such that $U \subseteq B$, and B can be written as $B \cup U$ where $U \subseteq \mathrm{B}$, then $w(\tau) \subseteq \{U \cup A: \quad U \in \tau, \quad A \subseteq X\}$.

Definition 1.21. A subset $A \subseteq X$ is called w-dense if $cl_w(A) = X$. We write $D_w(\tau)$ for the class of all w-dense set in X.

Theorem 1.22. $A \in D_w(\tau) \Leftrightarrow A \cap U \neq \emptyset$ for every nonempty $U \in \tau$.

Proof. Let $A \in D_w(\tau)$ with $A \cap U = \emptyset$ for some nonempty $U \in \tau$, so $A \subseteq X\backslash U$ and by (1.15) and (1.16) $cl_w(A) = X \subseteq X\backslash U$ and this contradicts $U \neq \emptyset$. The other side is clear using (1.20).

Corollary 1.23. If $\bigcap_{U_i \in \tau} U_i = \{a\}$ for some $a \in X$, then $cl_w(\{a\}) = X$.

Proof. By (1.22), take $A = \{a\}$.

Theorem 1.24. $A \subseteq X$ is w-dense $\Leftrightarrow$ A is dense in X, i.e. $D_w(\tau) = D(\tau)$.

Proof. By (1.15), $D_w(\tau) \subseteq D(\tau)$. Now, let $A \notin D_w(\tau)$, so $A \cap U = \emptyset$ for some $U \in \tau$, so $A \subseteq X\backslash U$ and $cl(A) \subseteq X\backslash U$ which shows that $A \notin D(\tau)$.

Theorem 1.25. Every clopen set in X is w-open and w-closed.

Proof. It is clear.

Corollary 1.26. The operations $int_w(\ )$ and $cl_w(\ )$ are dual to each other, that is,

1) $(int_w(A))^c = cl_w(A^c)$.
2) $(cl_w(A))^c = int_w(A^c)$.

2. w-Continues Function

In this section, we define and explore the notions of w-continuous, w-irresolute, and w-open (w-closed) functions within the framework of topological spaces. Additionally, we investigate their fundamental properties and provide various characterizations to enhance understanding of these functions.

Definition 2.1 [8,12,1,3]. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be continuous (κ-continuous, for $\kappa \in \{\alpha, \mathrm{semi}, \mathrm{pre}\}$) at $x \in X$ if for each $V \in \sigma$ containing $f(x)$, there is $U \in \tau$ (respectively, $U \in \kappa(\tau)$, for $\kappa \in \{\alpha, \mathrm{semi}, \mathrm{pre}\}$) containing x such that $f(U) \subseteq V$. A function f is continuous (κ- continuous) if it is continuous (κ-continuous) at every $x \in X$.

Theorem 2.2 [8,12,1,3]. A function $f : (X, \tau) \to (Y, \sigma)$ is

1) A continuous function if the inverse image of any open subset of Y is an open subset in X.
2) A κ-continuous function, for $\kappa \in \{\alpha, \mathrm{semi}, \mathrm{pre}\}$, if the image of any open subset of Y is an κ-open subset of X.

Definition 2.3. A function $f: (X, \tau) \longrightarrow (Y, \sigma)$ is called w-continuous at x if for every $V \in \sigma$ such that $f(x) \in V$, there exists $U \in w(\tau)$ with $x \in U$ satisfying $f(U) \subseteq V$. A function f is w-continuous if this property holds for all $x \in X$.

Proposition 2.4. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a function, then the next statements are equivalent:

1) f is a w-continuous function.
2) The preimage $f^{-1}(U)$ of every open subset U of Y is w-open in X.
3) The preimage $f^{-1}(F)$ of every closed subset F of Y is w-closed in X.

Proof. 1 ⇒ 2 Let U be an open set in Y with $x\in f^{-1}(U)$, then $f(x)\in U$, and there exists $W\in w(\tau)$ such that $x\in W$ and $f(W)\subseteq V$, so $W\subseteq f^{-1}(V)$. By (1.19) $f^{-1}(V)\in w(\tau)$.
2 ⇒ 3 Let F be a closed set in Y, so $U=Y\backslash F$ is an open set in Y, and so $f^{-1}(Y\backslash F)$ is an w-open set in X. Since $f^{-1}(Y\backslash F)=X\backslash f^{-1}(F)$, so $f^{-1}(F)$ is a w-closed set in X.
3 ⇒ 1 Let V be an open set in Y containing $f(x)$, so $F=Y\backslash V$ is a closed set in Y with $f(x)\notin F$. By (3) $U=X\backslash f^{-1}(V)\in \mathrm{w}(\tau)$ such that $x\in U$ and $f(U)\subseteq V$.

Theorem 2.5. Every continuous function is w-continuous.
Proof. It is obvious, since $\tau\backslash\{\emptyset\}\subseteq w(\tau)$.
The converse of the aforementioned result is not universally true, as demonstrated in the following example.
Example 2.6. Let $X=Y=\{1,2,3,4\}$ with $\tau=\{X,\emptyset,\{1\},\{2\},\{1,2\}\}$ and $\sigma=\{Y,\emptyset,\{1,2\},\{3,4\}\}$. Define a function $f:(X,\tau)\to(Y,\sigma)$ as the following: $f(1)=1$, $f(2)=3$, $f(3)=2$, $f(4)=4$, then f is a w-continuous, but it is not continuous.
Theorem 2.7. $f:X\to Y$ is a w-continuous function $\Leftrightarrow$ $f(cl_w(A))\subseteq cl(f(A))$.
Proof. Suppose f is a w-continuous function, so $f^{-1}(cl(f(A)))$ is a w-closed set in X which contains A. Since $cl_w(A)$ is the smallest w-closed set in X containing A, therefore $cl_w(A)\subseteq f^{-1}(cl(f(A)))$ and so $f(cl_w(A))\subseteq cl(f(A))$.
Now let, $(cl_w(A))\subseteq cl(f(A))$ and let $A=f^{-1}(E)$ where E is a closed set in Y, then $f\left(cl_w(f^{-1}(\mathrm{E}))\right)\subseteq cl\left(f(f^{-1}(E))\right)=\mathrm{cl(E)}=\mathrm{E}$, so $cl_w(f^{-1}(\mathrm{E}))\subseteq f^{-1}(\mathrm{E})$, and $f^{-1}(\mathrm{E})$ is w-closed set in X. Therefore f is w-continuous.
Theorem 2.8. $f:X\to Y$ is a w-continuous function $\Leftrightarrow int(f(A))\subseteq f(int_w(A))$.
Proof. It is clear.
Theorem 2.9. Let $f:X\to Y$ be a w-continuous function on X, and let $g:Y\to Z$ be a continuous function on Y, then the composition $g\circ f$ is a w-continuous function on X.
Proof. Since g is continuous, so $g^{-1}(V)$ is an open set in Y for every open set V in Z, and since f is a w-continuous function, so $f^{-1}(g^{-1}(V))=(g\circ f)^{-1}(V)$ is w-open in X. Therefore $g\circ f$ is w-continuous on X.
Theorem 2.10 [12]. If $f:X\to Y$ is α-continuous, then it is semi-continuous.
Theorem 2.11. Every semi-continuous is w-continuous.
Proof. It is obvious, since $\mathrm{semi}(\tau)\backslash\{\emptyset\}\subseteq w(\tau)$.
The converse of the aforementioned result is not universally true, as demonstrated in the following example.
Example 2.12. Let $X=\{1,2,3,4\}$, and let $\tau=\sigma=\{X,\emptyset,\{1\},\{2\},\{1,2\}\{2,3\}\{1,2,3\}\}$. Define a function $f:(X,\tau)\to(X,\sigma)$ as follows $f(1)=f(3)=1$, $f(2)=f(4)=2$, then f is w-continuous. Since $f^{-1}(\{1\})=\{1,3\}$ is not semi-open in X, so f is not semi-continuous.
Definition 2.13[11]. $f:X\to Y$ is called an irresolute function if $f^{-1}(V)$ is semi-open in X for any semi-open set V in Y.
Definition 2.14[18, 15]. $f:X\to Y$ is called an κ-irresolute function if $f^{-1}(V)$ is κ-open in X for any κ-open set V in Y, where $\kappa\in\{\alpha,\mathrm{pre}\}$.
Definition 2.15. $f:X\to Y$ is called an w-irresolute function if for every w-open set V⊆Y, the preimage $f^{-1}(V)$ is w-open in X.
Proposition 2.16. For a function $f:(X,\tau)\longrightarrow(Y,\sigma)$, the following statements are equivalent:

1) f is w-irresolute.
2) For each $V\in \mathrm{w}(\sigma)$ containing $f(x)$, there exists $U\in w(\tau)$ containing x such that $f(U)\subseteq V$.
3) The preimage of every w-closed set in Y is w-closed in X.

Proof. (1 ⇒ 2) Let $V\in w(\sigma)$ with $f(x)\in V$, since f is w-irresolute, so $f^{-1}(V)\in w(\tau)$ and $x\in f^{-1}(V)$. Put $U=f^{-1}(V)$, then $x\in$U and $f(V)\subseteq U$.
(2 ⇒ 3) Let E be a w-closed set in Y and let $x\notin f^{-1}(E)$, so $V=Y\backslash E$ is an w-open set in Y, and $f(x)\in V$. Then there exists $U\in w(\tau)$ containing x such that $f(U)\subseteq V$, so $U\subseteq f^{-1}(Y\backslash E)=X\backslash f^{-1}(E)$ is w-open in X, and $f^{-1}(E)$ is w-closed in X.
(3 ⇒ 1) It is obvious.

Theorem 2.17. $f: X \to Y$ is an w-irresolute function $\Leftrightarrow$ $f(cl_w(A)) \subseteq cl_w(f(A))$, for every $A \subseteq X$.
Proof. Assume that f is an w-irresolute function , since $cl_w(f(A))$ is w-closed in Y containing $f(A)$, so $f^{-1}(cl_w(f(A)))$ is w-closed in X containing A, therefore $cl_w(A) \subseteq f^{-1}(cl_w(f(A)))$, and consequently, $f(cl_w(A)) \subseteq cl_w(f(A))$.
On the other side, let $f(cl_w(A)) \subseteq cl_w(f(A))$ for any $A \subseteq X$, and let E be a w-closed subset of Y, then $f\left(cl_w(f^{-1}(\mathrm{E}))\right) \subseteq cl_w\left(f(f^{-1}(E))\right) \subseteq \mathrm{cl}_w(\mathrm{E}) = E$, so $cl_w(f^{-1}(\mathrm{F})) \subseteq f^{-1}(\mathrm{F})$, and $f^{-1}(\mathrm{E})$ is w-closed set in X. Therefore f is w-irresolute.
Theorem 2.18. $f: X \to Y$ is an w-irresolute function $\Leftrightarrow$ following condition holds for every $A \subseteq Y$

$$cl_w(f^{-1}(A)) \subseteq f^{-1}(cl_w(A)).$$

Proof. Assume that f is w-irresolute, since $f^{-1}(cl_w(A)$ is w-closed in X, and $f^{-1}(A) \subseteq f^{-1}(cl_w(A)$, then $cl_w(f^{-1}(A)) \subseteq f^{-1}(cl_w(A))$.
Now let, A be a w-closed set in Y, then $cl_w(f^{-1}(A)) \subseteq f^{-1}(cl_w(A)) = f^{-1}(A)$, so $f^{-1}(A) = cl_w(f^{-1}(A))$, and this shows that f is w-irresolute.
Theorem 2.19[18]. If $f : X \to Y$ is an α- irresolute function, then it is α- continuous.
Theorem 2.20. If $f : X \to Y$ is a continuous function, then it is w-irresolute.
Proof. Suppose that A is w-open in Y, then there is an open set in Y such that $U \subseteq A$, so $f^{-1}(U)$ is an open set in X and $f^{-1}(U) \subseteq f^{-1}(A)$, hence $f^{-1}(A)$ is a w-open set in X.
The converse of the above theorem is not necessarily true as illustrated by the following example.
Example 2.21. Let $(\mathbb{R}, \tau)$ be the usual topology space, and $\tau^* = \{U \cup \{0\}: U \in \tau\}$, and let (Y, σ) be the sierpinski space. i.e. $Y = \{0,1\}$ and $\sigma = \{Y, \emptyset, \{0\}\}$. Defined a function $f: (X, \tau^*) \longrightarrow (Y, \sigma)$ by $f(x) = 0$ for rational number x and $f(x) = 1$ for irrational number x. Then f is w-irresolute, but it is not continuous.
Theorem 2.22. If $f : X \to Y$ an w-irresolute function, then it is w-continuous.
Proof. It is obvious, by using (1.4).
Theorem 2.23. If $f : X \to Y$ is a w- continuous function, then it is w- irresolute.
Proof. Suppose that A is w-open in Y, then there is an open set V in Y such that $V \subseteq A$, so $f^{-1}(V)$ is an w-open set in X and $f^{-1}(V) \subseteq f^{-1}(A)$. By (1.19) $f^{-1}(A)$ is also w-open in X and this shows that f is w-irresolute

Remark 2.24. The above results show that the concepts of w- continuity and w- irresoluteness of functions are coincided.
Theorem 2.25. If $f: X \to Y$ and $g: Y \to Z$ are w-irresolute functions, then the composition $g \circ f$ is also w-irresolute function.
Proof. Let E be an w-closed set in Z, since g is an w-irresolute, then $g^{-1}(E)$ is an w-closed set in Y and since f is w-irresolute, so $f^{-1}(g^{-1}(E)) = (g \circ f)^{-1}(E)$ is w-closed in X. Thus $g \circ f$ is w-irresolute.
Theorem 2.26. If $f: X \to Y$ is an w-irresolute function, and $g: Y \to Z$ is a w-continuous function, then $g \circ f$ is w-continuous.
Proof. It is obvious.
Definition 2.27[8]. $f : X \to Y$ is called an open(closed) function if $f(U)$ is open(closed) in Y for any open set U of X.
Definition 2.28[12,14,3)]. $f : X \to Y$ is called an κ –open (κ –closed) function if f (U) is κ –open (κ -closed) in Y for any open (closed) set U in X, where $\kappa \in \{\alpha, \text{semi}, \text{pre}\}$.
Definition 2.29. $f : X \to Y$ is called an w-open(w-closed) function if the image of any open(closed) set in X is an w-open(w-closed) set in Y.

Theorem 2.30[12,14,3]. If $f : X \to Y$ is an open (closed) function, then it is κ –open (κ –closed), where $\kappa \in$\{α, semi , pre \}
Theorem 2.31[12]. If $f : X \to Y$ is an α-open(α-closed) function, then it is semi-open (semi-closed).
Theorem 2.32. Let $f : X \to Y$ be a function between two topological spaces. The following statements hold:

1) If f is an open function, then f is w-open.
2) If f is a closed function, then f is w-closed.

Proof. It is obvious, by using (1.4).

Theorem 2.33. If $f: X \to Y$ is a semi-open (semi-closed) function, then f is w-open (w-closed).
Proof. It is obvious, by using (1.4).

Consider a specific example where the conditions for being semi-open are not met, even though the function may still be open. This counterexample demonstrates that the converse does not hold in general.

Example 2.34 Let $X = \{1, 2, 3, 4\}$, and let $\tau = \{X, \emptyset, \{1, 2\}\}$ and $\sigma = \{X, \emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}, \{2, 3, 4\}\}$. Defined a function $f: (X, \tau) \to (X, \sigma)$ by $f(x) = x$, for all $x \in X$. It is clear that f is a w-open ma.

Since $f(\{1, 2\}) = \{1, 2\}$, so f is not semi-open and so it is not open.

Theorem 2.35. If $f: X \to Y$ is an open(closed) function, and $g: Y \to Z$ is an w-open(w-closed) function, then the composition $g \circ f$ is an w-open(w-closed) function.

Proof. It is obvious.

Theorem 2.36. Let $f: (X, \tau) \to (Y, \sigma)$ be a bijective w-closed function, and let U be any open subset of X such that $f^{-1}(A) \subseteq U$ where $A \subseteq Y$, then there is an w-open set $V \subseteq Y$ such that $A \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof. Let $V = Y \backslash f(X \backslash U)$, it is clear that V is an w-open set in Y and it satisfies the required conditions.

Theorem 2.37 Let $f: (X, \tau) \to (Y, \sigma)$ be a bijective w-open function, and let F be any closed subset of X such that $f^{-1}(A) \subseteq F$ where $A \subseteq Y$, then there exists w-closed set E in Y such that $A \subseteq E$ and $f^{-1}(E) \subseteq F$.

Proof. Similar to (2.36), where $V = Y \backslash f(X \backslash F)$.

Theorem 2.38. $f: X \to Y$ is an w-open function $\Leftrightarrow f\big(int(A)\big) \subseteq int_w\big(f(A)\big)$, for any $A \subseteq X$.

Proof. Assume that f is w-open, since $f(int(A)) \subseteq f(A)$ and $f(int(A))$ is an w-open set in Y for any $A \subseteq X$, then $f\big(int(A)\big) = int_w(f\big(int(A)\big)) \subseteq int_w f(A)$.

On the other side, let $f(int(A)) \subseteq int_w(f(A))$ for any $A \subseteq X$, and let V be an open set in X, so $f(V) = f(int(V)) \subseteq int_w(f(V))$, and this shows that $f(V)$ is w-open in Y. Therefore f is w-open.

Theorem 2.39. $f: X \to Y$ is a w-closed function $\Leftrightarrow cl_w(f(A)) \subseteq f(cl(A))$ for any $A \subseteq X$.

Proof. Assume that f is w-closed, since $f(A) \subseteq f(cl(A))$, so $cl_w(f(A)) \subseteq cl_w(f(cl(A))) = f(cl(A))$.

Now let, $cl_w(f(A)) \subseteq f(cl(A))$ for any $A \subseteq X$, and let F be a closed subset of X, then $f(F) \subseteq cl_w(f(F)) \subseteq f(cl(F)) = f(F)$, and this shows that $f(F)$ is a w-closed set in Y. Therefore f is w-closed.

The next theorem presents the necessary and the sufficient condition for a function to be both w-open and w-closed simultaneously.

Theorem 2.40 If $f: X \to Y$ is bijective, then f is w-open $\Leftrightarrow f$ is w-closed.

Proof. It is obvious.

Theorem 2.41. Let $g \circ f: X \to Z$ be an w-open(w-closed) function, where $f: X \to Y$ and $g: Y \to Z$. Then

1) If f is continuous and bijective, then g is w-open(w-closed).
2) If g is w-irresolute, bijective, then f is w-open (w-closed).

Proof.

1) If V is an arbitrary open set in Y, so $f^{-1}(V)$ is open in X, and so $g \circ f(f^{-1}(V)) = g(V)$ is w-open in Z. This implies that g is w-open.
2) If U is an arbitrary open set in X, so $g \circ f(U)$ is w-open in Z, and so $g^{-1}\big((g \circ f)(U)\big) = f(U)$ is w-open in Y. This implies that f is w-open.

Conclusion

The research presents a new classification of some generalized open sets, namely w-open (w-closed) sets, and establishes new types of functions, such as w-continuous, w-irresolute, w-open, and w-closed functions. It has been found that there are significant relationships among these function types, demonstrating the intricate relationships that exist when applying these new definitions. This research enhances the understanding of how these functions interact, providing insights into their roles in topological spaces and paving the way for further investigations in the field of topology.

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