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Generalizations of harmonic starlike functions defined by new generalized derivative operator with respect to symmetric points

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التعميمات الخاصة بالدوال النجمية التوافقية معرفة بواسطة مؤثر تفاضلي جديد بالنسبة للنقاط المتماثلة

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Abstract:
In this paper, we define the operator $\mathcal{D}^{m,q,s}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]$, and applying this operator to the harmonic function. Using this operator we introduce a new class of complex-valued harmonic functions with respect to symmetric points. We obtain coefficient bounds, extreme points, distortion bounds, convex combinations, and inclusion results and closure under an integral operator for this family of harmonic univalent functions.

Keywords Harmonic functions, Dziok-Srivastava operator, derivative operator, symmetric point, integral operator.

الملخص
في هذا البحث، نقوم بتعريف المؤثر $\mathcal{D}^{m,q,s}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]$ وتطبيقه على الدوال التوافقية. باستخدام هذا المؤثر، نقدم فئة جديدة من الدوال التوافقية ذات القيم المركبة المرتبطة بالنقاط المتماثلة. تتضمن النتائج التي توصلنا إليها حدود معاملات، نقاط القصوى، حدود التشوه، التراكيب المحدبة، ونتائج الاحتواء، بالإضافة إلى إثبات الانغلاق تحت المؤثر التكاملي لهذه الفئة من الدوال التوافقية الأحادية.

الكلمات المفتاحية: الدوال التوافقية، مؤثر Dziok-Srivastava، المؤثر التفاضلي، النقاط المتماثلة، المؤثر التكاملي.

1. Introduction

We denote by S_H the family of functions

$$f = h + \overline{g}, \tag{1.1}$$

that are harmonic and sense-preserving in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in S_H$ we may write the analytic functions h and g in the forms

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad (0 \le b_1 < 1). \tag{1.2}$$

We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in S_H is that $|h'(z)| > |g'(z)|$ in S_H. See Clunie and Sheil-Small [5].

Hence

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}, \quad (0 \le b_1 < 1). \tag{1.3}$$

We denote $S_{\overline{H}}$ as the subclass of S_H comprising harmonic functions $f = h + \overline{g}$ given by (1.1) of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}, \quad (0 \le b_1 < 1). \tag{1.4}$$

For parameters $\alpha_i \in \mathbb{C}(i = 1, \ldots, q)$, and $\beta_j \in \mathbb{C}\backslash\{0, -1, -2, \ldots.\}(j = 1, \ldots, s)$, the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ is defined as:

$${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k, \ldots, (\alpha_q)_k}{(\beta_1)_k, \ldots, (\beta_s)_k} \frac{z^k}{k!},$$

$(q \le s + 1, q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{U})$, and $(\nu)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(\nu)_k = \frac{\Gamma(\nu + k)}{\Gamma(k)} = \begin{cases} 1, & k = 0, \nu \in \mathbb{C} \setminus \{0\}; \\ \nu(\nu + 1)(\nu + 2)\ldots.(\nu + k - 1), & k \in \mathbb{N} = \{1,2,3,\ldots\}. \end{cases}$$

Dziok and Srivastava [8] defined the linear operator

$$H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)\phi(z) = z + \sum_{k=2}^{\infty} \Upsilon_k a_k z^k, \tag{1.5}$$

Where

$$\Upsilon_k = \frac{(\alpha_1)_{k-1}, \ldots, (\alpha_q)_{k-1}}{(\beta_1)_{k-1}, \ldots, (\beta_s)_{k-1}(k-1)!}, \tag{1.6}$$

for convenience we write

$$H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)\phi(z) = H^{q,s}[\alpha_1]\phi(z).$$

Recently, the authors in [17] presented a function $\mathcal{M}^{\mu}_{\lambda_1,\lambda_2,\ell,d}$ defined as follows

$$\mathcal{M}^{m}_{\lambda_1,\lambda_2,\ell,d}(z) = z + \sum_{k=2}^{\infty} \Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell) z^k, \tag{1.7}$$

where

$$\Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell) = \left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m, \tag{1.8}$$

$m, d \in \mathbb{N}_0 = \{0,1,2, \ldots\}$, $\lambda_2 \ge \lambda_1 \ge 0, \ell \ge 0$, and $\ell + d > 0$.

Utilizing the Hadamard product, the linear operator $\mathcal{D}^{m,q,s}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j]$ is defined as follows

$$\mathcal{D}^{m,q,s}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j]\phi(z) = \mathcal{M}^{m}_{\lambda_1,\lambda_2,\ell,d}(z) * H^{q,s}[\alpha_1]\phi(z),$$

then, from (1.5) and (1.7), we have

$$\mathcal{D}^{m,q,s}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j]\phi(z) = z + \sum_{k=2}^{\infty} \Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell)\Upsilon_k a_k z^k, \tag{1.9}$$

where Υ_k and $\Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell)$ are defined in (1.6) and (1.8), respectively.

Now, by applying the operator defined in (1.9) to the harmonic function given by (1.1),

we get

$$\mathcal{D}^{m,q,s}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j]f(z) = \mathcal{D}^{m,q,s}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j]h(z) + \overline{\mathcal{D}^{m,q,s}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j]g(z)}. \tag{1.10}$$

We note that, the operator $\mathcal{D}^{m,q,s}_{\lambda_1,0,1,d}[\alpha_i, \beta_j]f(z) = I^{m,\ell}_{q,s,\lambda}(\alpha_1, \beta_1)f(z)$ was studied by El-Ashwah and Aouf (see[9]).

Also, if the co-analytic part $g = 0$, then $\mathcal{D}^{0,q,s}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z) = \mathcal{D}^{m,q,s}_{\lambda_1,\lambda_2,0,1}[\alpha_i,\beta_j]f(z) = H^{q,s}[\alpha_1]f(z)$ was studied by Dziok-Srivastava (see [8]). Also we note that

(i) For $q = s+1, \alpha_i = 1(i = 1,\cdots,s+1)$,$\beta_j = 1(j = 1,\cdots,s), \lambda_2 = 0$ and $\ell = 1$ we get the operator $I^m(\lambda,\ell)$ given by Cătas [4].

(ii) For $q = s+1, \alpha_i = 1(i = 1,\cdots,s+1)$,$\beta_j = 1(j = 1,\cdots,s), \lambda_2 = d = 0$ and $\lambda_1 = \ell = 1$ we get salagean operator D^m (see [18]).

(iii) For $q = s+1, \alpha_i = 1(i = 1,\cdots,s+1), \beta_j = 1(j = 1,\cdots,s)$,$\lambda_2 = d = 0$ and $\ell = 1$ we get the operator D^m_λ given by Al-Oboudi [1].

(iv) For $q = 2, s = 1, \alpha_1 = n+1, \alpha_2 = 1$ and $\beta_1 = 1$ we get derivative operator $\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}$ given by Oshah and Darus [17].

(v) For $q = 2, s = 1, \alpha_1 = \alpha+1, \alpha_2 = 1, \beta_1 = 1$,$\ell = 1$ and $d = 0$, we get derivative operator $D^{m,\alpha}_{\lambda_1,\lambda_2}$ given by Eljamal and Darus [11].

(vi) For $q = 2, s = 1, \alpha_1 = \delta+1, \alpha_2 = 1, \beta_1 = 1$ and $\ell = 1$, we get derivative operator $D^{m,b}_{\lambda_1,\lambda_2,\delta}$ given by El-Yagubi and Darus [10].

(vii) For $q = 2, s = 1, \alpha_1 = n+1, \alpha_2 = 1, \beta_1 = 1, \lambda_1 = 1$ and $\lambda_2 = 0$, we get derivative operator $I^m_{\alpha,\beta}$ given by Swamy [20].

(viii) For $q = s+1$, $\lambda_2 = 0$ and $\lambda_1 = \ell = 1, d = \lambda$, we get derivative operator I^m_λ given by Cho and Srivastava [6].

(ix) For $q = 2, s = 1, \alpha_1 = n+1, \alpha_2 = 1, \beta_1 = 1$ and $\lambda_2 = d = 0, \ell = 1$, we get derivative operator D^λ_n given by Darus and Al-Shaqsi [12].

(x) For $q = s+1$ and $\lambda_2 = 0, \lambda_1 = \ell = d = 1$, we get derivative operator L^mgiven by Uralegaddi and Somanatha [21].

By suitably specializing the values of $q, s, \alpha_i(i = 1,\cdots,q)$ and $\beta_j(j = 1,\cdots,s)$, we obtain

$$\mathcal{D}^{m,2,1}_{\lambda_1,\lambda_2,\ell,d}[n+1,1;1]f(z) = \mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z) = z + \sum_{k=2}^{\infty} \Lambda^{m,k}_d(\lambda_1,\lambda_2,\ell)\frac{(n+1)_{k-1}}{(1)_{k-1}}a_k z^k + \sum_{k=1}^{\infty} \Lambda^{m,k}_d(\lambda_1,\lambda_2,\ell)\frac{(n+1)_{k-1}}{(1)_{k-1}}\overline{b_k z^k}, n > -1,$$

$$\mathcal{D}^{m,2,1}_{\lambda_1,\lambda_2,\ell,d}[a,1;c]f(z) = \mathcal{D}^{m}_{\lambda_1,\lambda_2,\ell,d}[a,c]f(z) = z + \sum_{k=2}^{\infty} \Lambda^{m,k}_d(\lambda_1,\lambda_2,\ell)\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^k + \sum_{k=1}^{\infty} \Lambda^{m,k}_d(\lambda_1,\lambda_2,\ell)\frac{(a)_{k-1}}{(c)_{k-1}}\overline{b_k z^k}, (a \in \mathbb{R}, c \in \mathbb{R}\backslash\mathbb{Z}_0).$$

Motivated by earlier works of [2,9,13,14,22,23] on harmonic functions, we introduce a new class $HS^{m,q,s}_{S^*}(\lambda_1,\lambda_2,\ell,d,[\alpha_i,\beta_j],\delta)$ of S_H that are starlike with respect to symmetric points.

Definition 1.1 *For* $0 \le \delta < 1$ *and* $z = re^{i\theta} \in \mathbb{U}$, *we let* $HS^{m,q,s}_{S^*}(\lambda_1,\lambda_2,\ell,d,[\alpha_i,\beta_j],\delta)$ *a subclass of* S_H *of the form* $f = h + \overline{g}$ *given by (1.3) and satisfying the analytic criteria*

$$\Re e\left\{\frac{2z(\mathcal{D}^{m,q,s}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z))'}{z'\left[\mathcal{D}^{m,q,s}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z) - \mathcal{D}^{m,q,s}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(-z)\right]}\right\} > \delta\,, \quad (1.11)$$

where $\mathcal{D}^{m,q,s}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)$ *is defined by (1.10) and* $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$.

Also, we let $\overline{HS}^{m,q,s}_{S^*}(\lambda_1,\lambda_2,\ell,d,[\alpha_1],\delta) = HS^{m,q,s}_{S^*}(\lambda_1,\lambda_2,\ell,d,[\alpha_i,\beta_j],\delta) \cap S_{\overline{H}}$.

By suitably specializing the values of $q, s, m, \lambda_1, \lambda_2, \ell, d, \alpha_i$ and β_j, the class $\overline{HS}^{m,q,s}_{S^*}(\lambda_1,\lambda_2,\ell,d,[\alpha_i,\beta_j],\delta)$ leads to various subclasses which were studied by various authors as follows

(i) $\overline{HS}^{0,q,s}_{S^*}(\lambda_1,0,1,d,[\alpha_i,\beta_j],\delta) = \overline{HS}^{\lambda,\ell,m}_{S^*}(q,s,[\alpha_1,\beta_1],\gamma)$ which was studied by R.El-Ashwah et al. [9],

(ii) $\overline{HS}^{0,q,s}_{S^*}(\lambda_1,\lambda_2,\ell,d,[\alpha_i,\beta_j],\delta) = \overline{HS}_{S^*}([\alpha_1],\gamma)$ which was studied by Murugusundaramoorthy et al. [16],

(iii) $\overline{HS}_{S^*}^{m,2,1}(1,0,1,0,[1,1],\delta) = \overline{SH}_s(m,\alpha)$ which were studied by AL-Khal and AlKharsani [3].

And, we note that

- $\overline{HS}_{S^*}^{m,2,1}(\lambda_1, 0,1, d, [1,1;1], \delta) = \overline{HS}_{s^*}(\lambda, \ell, \delta)$

$$= \left\{ f(z) \in \overline{H} \colon \Re e \left(\frac{2z(I^m(\lambda,\ell)f(z))'}{z'[I^m(\lambda,\ell)f(z) - I^m(\lambda,\ell)f(-z)]} \right) > \delta \right\}.$$

- $\overline{HS}_{S^*}^{m,2,1}(1,0,1, d, [1,1;1], \delta) = \overline{HS}_{s^*}^{m}(\lambda, \delta)$

$$= \left\{ f(z) \in \overline{H} \colon \Re e \left(\frac{2z(I_\lambda^m f(z))'}{z'[I_\lambda^m f(z) - I_\lambda^m f(-z)]} \right) > \delta \right\}.$$

- $\overline{HS}_{S^*}^{m,2,1}(\lambda_1, \lambda_2, \ell, d, [n+1,1;1], \delta) = \overline{HS}_{s^*}^{m,n}(\lambda_1, \lambda_2, \ell, d, \delta)$

$$= \left\{ f(z) \in \overline{H} \colon \Re e \left(\frac{2z\left(\mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)\right)'}{z'\left[\mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z) - \mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(-z)\right]} \right) > \delta \right\}.$$

- $\overline{HS}_{S^*}^{m,2,1}(\lambda_1, \lambda_2, 1,0, [\alpha+1,1;1], \delta) = \overline{HS}_{s^*}^{m,\alpha}(\lambda_1 \lambda_2, \delta)$

$$= \left\{ f(z) \in \overline{H} \colon \Re e \left(\frac{2z\left(D_{\lambda_1,\lambda_2}^{m,\alpha} f(z)\right)'}{z'\left[D_{\lambda_1,\lambda_2}^{m,\alpha} f(z) - D_{\lambda_1,\lambda_2}^{m,\alpha} f(-z)\right]} \right) > \delta \right\}.$$

- $\overline{HS}_{S^*}^{m,2,1}(\lambda_1, \lambda_2, \ell, d, [a,1;c], \delta) = \overline{HS}_{s^*}^{m,\ell,d}(\lambda_1, \lambda_2, a, c, \delta)$

$$= \left\{ f(z) \in \overline{H} \colon \Re e \left(\frac{2z\left(\mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m}[a,c] f(z)\right)'}{z'\left[\mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m}[a,c] f(z) - \mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m}[a,c] f(-z)\right]} \right) > \delta \right\}.$$

Observe that, the class $HS_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$ reduce to the class $S_s^*(\delta)$ of starlike functions concerning symmetric points as introduced by Sakaguchi [19] if $m = 0, q = s+1, \alpha_i = 1 (i = 1, \cdots, s+1)$,$\beta_j = 1 (j = 1, \cdots, s)$, and the co-analytic part of $f = h + \overline{g}$ is identically zero ($g \equiv 0$). Moreover, the class $HS_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$ reduce to the class $K_s(\delta)$ of convex functions concerning symmetric points, as introduced by Das and Singh [7] if $q = s+1, \alpha_i = 1 (i = 1, \cdots, s+1)$,$\beta_j = 1 (j = 1, \cdots, s), \lambda_2 = d = 0, m = \lambda_1 = \ell = 1$ and the co-analytic part of $f = h + \overline{g}$ is identically zero ($g \equiv 0$).

This paper presents the derivation of coefficient conditions for the classes $HS_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$ and $\overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$. Additionally, it establishes a representation theorem, investigates inclusion properties and distortion bounds, and demonstrates inclusion results as well as closure under an integral operator for the class $\overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$.

2 Coefficient characterization

Throughout this paper, unless otherwise specified, we assume $q, s \in \mathbb{N}, a_1 = 1, \alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_s \in \mathbb{R}^+$ and $0 \le \delta < 1$. We start by establishing a sufficient condition for functions f belonging to the class $HS_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$.

Theorem 2.1 *Let $f = h + \overline{g}$ be given by (1.3). Furthermore we let*

$$\sum_{k=2}^{\infty} \frac{[2k + \delta((-1)^k - 1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell) \Upsilon_k |a_k| + \sum_{k=1}^{\infty} \frac{[2k - \delta((-1)^k - 1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell) \Upsilon_k |b_k| \le 1, \quad (2.1)$$

where Υ_k and $\Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell)$ are defined in (1.6) and (1.8), respectively. Then f is sense-preserving, harmonic univalent in $\mathbb{U}$ and and belongs to the class $HS_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$.

Proof. According to the condition (1.11), we only need to show that if the inequality (2.1) holds, then

$$\Re e\left\{\frac{2z(\mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m,q,s}[\alpha_i,\beta_j]f(z))'}{z'\left[\mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m,q,s}[\alpha_i,\beta_j]f(z)-\mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m,q,s}[\alpha_i,\beta_j]f(-z)\right]}\right\} = \Re e\,\frac{A(z)}{B(z)} > \delta,$$

where

$$A(z) = 2z(\mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m,q,s}[\alpha_i,\beta_j]f(z))'$$

$$= 2z'\left[z + \sum_{k=2}^{\infty} k\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k a_k z^k - \sum_{k=1}^{\infty} k\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k \overline{b_k z^k}\right],$$

and

$$B(z) = z'\left[\mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m,q,s}[\alpha_i,\beta_j]f(z) - \mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m,q,s}[\alpha_i,\beta_j]f(-z)\right]$$

$$= z'[2z - \sum_{k=2}^{\infty} [(-1)^k - 1]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k a_k z^k - \sum_{k=1}^{\infty} [(-1)^k - 1]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k \overline{b_k z^k}].$$

Using the condition $\Re e(\omega(z)) > \delta \Leftrightarrow |1-\delta+\omega| > |1+\delta-\omega|$, it is sufficient to prove that

$$|A(z) + (1-\delta)B(z)| - |A(z) - (1+\delta)B(z)| > 0. \qquad (2.2)$$

Substituting the expressions for $A(z)$ and $B(z)$ from (2.1) into (2.2), we obtain

$$\Big|2(2-\delta)z + \sum_{k=2}^{\infty} [2k - (1-\delta)((-1)^k - 1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k a_k z^k$$

$$- \sum_{k=1}^{\infty} [2k + (1-\delta)((-1)^k - 1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k \overline{b_k z^k}\Big|$$

$$-\Big|-2\delta z + \sum_{k=2}^{\infty} [2k + (1-\delta)((-1)^k - 1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k a_k z^k$$

$$- \sum_{k=1}^{\infty} [2k - (1+\delta)((-1)^k - 1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k \overline{b_k z^k}\Big|$$

$$\geq 4(1-\delta)|z| - 2\sum_{k=2}^{\infty} [2k + \delta((-1)^k - 1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k |a_k||z^k|$$

$$-2\sum_{k=1}^{\infty} [2k - \delta((-1)^k - 1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k |b_k||z^k|$$

$$= 4(1-\delta)|z|[1 - \sum_{k=2}^{\infty} \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)}\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k |a_k||z|^{k-1}$$

$$- \sum_{k=1}^{\infty} \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)}\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k |b_k||z|^{k-1}]$$

$$\geq 4(1-\delta)[1 - \sum_{k=2}^{\infty} \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)}\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k |a_k|$$

$$- \sum_{k=1}^{\infty} \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)}\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k |b_k|]$$

This final expression is non-negative by (2.1). The harmonic univalent functions

$$f(z) = z + \sum_{k=2}^{\infty} \frac{2(1-\delta)}{[2k+\delta((-1)^k-1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k} X_k z^k$$

$$+ \sum_{k=1}^{\infty} \frac{2(1-\delta)}{[2k-\delta((-1)^k-1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k} \overline{Y_k}\overline{z^k}. \qquad (2.3)$$

where $\sum_{k=2}^{\infty}|X_k|+\sum_{k=1}^{\infty}|Y_k|=1$, show that the coefficient bound given by (2.1) is sharp. The functions f of the form (2.3) are in $HS_{S^*}^{m,q,s}(\lambda_1,\lambda_2,\ell,d,[\alpha_i,\beta_j],\delta)$ because

$$\sum_{k=2}^{\infty}\frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)}\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k|a_k|$$

$$+\sum_{k=1}^{\infty}\frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)}\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k|b_k|=\sum_{k=2}^{\infty}|X_k|+\sum_{k=1}^{\infty}|Y_k|=1.$$

The proof is now complete.

By setting $\lambda_2=0,\ell=1$ we derive the following corollary

Corollary 2.2 *[9] Let $f=h+\overline{g}$ be given by (1.1). Furthermore we let*

$$\sum_{k=2}^{\infty}\frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Upsilon_k|a_k|$$

$$+\sum_{k=1}^{\infty}\frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Upsilon_k|b_k|\le 1,$$

where Υ_k is defined in *(1.6)*. Then f is sense-preserving, harmonic univalent in $\mathbb{U}$ and $f\in HS_{S^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\delta)$.

We proceed to prove the following theorem, which states that condition (2.1) is also necessary for functions f to belong to the class $\overline{HS}_{S^*}^{m,q,s}(\lambda_1,\lambda_2,\ell,d,[\alpha_i,\beta_j],\delta)$.

Theorem 2.3 *Let $f=h+\overline{g}$ be given by (1.4). Then $f\in\overline{HS}_{S^*}^{m,q,s}(\lambda_1,\lambda_2,\ell,d,[\alpha_i,\beta_j],\delta)$ if and only if*

$$\sum_{k=2}^{\infty}\frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)}\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k|a_k|$$

$$+\sum_{k=1}^{\infty}\frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)}\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k|b_k|\le 1, \tag{2.4}$$

where Υ_k and $\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)$ are defined in *(1.6)* and *(1.8)*, respectively.

Proof. Since $\overline{HS}_{S^*}^{m,q,s}(\lambda_1,\lambda_2,\ell,d,[\alpha_i,\beta_j],\delta)\subset HS_{S^*}^{m,q,s}(\lambda_1,\lambda_2,\ell,d,[\alpha_i,\beta_j],\delta)$, Our task is limited to proving the the "only if" part of the theorem. To accomplish this, for functions f of the form (1.4), we observe that the condition.

$$\Re e\left\{\frac{2z(\mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m,q,s}[\alpha_i,\beta_j]f(z))'}{z'\left[\mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m,q,s}[\alpha_i,\beta_j]f(z)-\mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m,q,s}[\alpha_i,\beta_j]f(-z)\right]}\right\}>\delta,$$

is equivalent to

$$\Re e\left\{\frac{2(1-\delta)-\sum_{k=2}^{\infty}[2k+\delta((-1)^k-1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k a_k z^{k-1}-\frac{\overline{z}}{z}\sum_{k=1}^{\infty}[2k-\delta((-1)^k-1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k b_k\overline{z}^{k-1}}{2+\sum_{k=2}^{\infty}((-1)^k-1)\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k a_k z^{k-1}-\frac{\overline{z}}{z}\sum_{k=1}^{\infty}((-1)^k-1)\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k b_k\overline{z}^{k-1}}\right\}>0\,. \tag{2.5}$$

The required condition (2.5) must be satisfied for all values of $z\in\mathbb{U}$. By selecting values of z on the positive real axis where $0\le z=r<1$, the condition must hold

$$\Re e\left\{\frac{2(1-\delta)-\sum_{k=2}^{\infty}[2k+\delta((-1)^k-1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k a_k r^{k-1}-\sum_{k=1}^{\infty}[2k-\delta((-1)^k-1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k b_k r^{k-1}}{2+\sum_{k=2}^{\infty}((-1)^k-1)\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k a_k r^{k-1}-\sum_{k=1}^{\infty}((-1)^k-1)\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_s^q b_k r^{k-1}}\right\}>0. \tag{2.6}$$

If the condition (2.4) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Hence there exists $z_0=r_0$ in (0,1) for which the quotient in (2.6) is negative. This contradicts the required condition for $f(z)\in\overline{HS}_{S^*}^{m,q,s}(\lambda_1,\lambda_2,\ell,d,[\alpha_i,\beta_j],\delta)$ and so the proof of Theorem 2.3 is complete .

By appropriately specifying the values of $\lambda_1, \lambda_2, \ell, d, q, s, \alpha_i (i = 1, \cdots, q)$ and $\beta_j (j = 1, \cdots, s)$, the following corollaries are derived

Corollary 2.4 *For* $f = h + \overline{g} \in \overline{HS}_{S^*}^{\lambda,\ell,m}(q, s, [\alpha_1, \beta_1], \gamma)$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Upsilon_k |a_k|$$

$$+\sum_{k=1}^{\infty} \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Upsilon_k |b_k| \le 1$$

Corollary 2.5 *For* $f = h + \overline{g} \in \overline{HS}_{S^*}(\lambda, \ell, \delta)$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m |a_k|$$

$$+\sum_{k=1}^{\infty} \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m |b_k| \le 1.$$

Corollary 2.6 *For* $f = h + \overline{g} \in \overline{HS}_{S^*}^{m}(\lambda, \delta)$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)} \left[\frac{k+\ell}{1+\ell}\right]^m |a_k|$$

$$+\sum_{k=1}^{\infty} \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)} \left[\frac{k+\ell}{1+\ell}\right]^m |b_k| \le 1$$

Corollary 2.7 *For* $f = h + \overline{g} \in \overline{HS}_{S^*}^{m,n}(\lambda_1, \lambda_2, \ell, d, \delta)$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell) \frac{(n+1)_{k-1}}{(1)_{k-1}} |a_k|$$

$$+\sum_{k=1}^{\infty} \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell) \frac{(n+1)_{k-1}}{(1)_{k-1}} |b_k| \le 1.$$

Corollary 2.8 *For* $f = h + \overline{g} \in \overline{HS}_{S^*}^{m,\alpha}(\lambda_1 \lambda_2, \delta)$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)} \left[\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)}\right]^m \frac{(\alpha+1)_{k-1}}{(1)_{k-1}} |a_k|$$

$$+\sum_{k=1}^{\infty} \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)} \left[\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)}\right]^m \frac{(\alpha+1)_{k-1}}{(1)_{k-1}} |b_k| \le 1.$$

Corollary 2.9 *For* $f = h + \overline{g} \in \overline{HS}_{S^*}^{m,\ell,d}(\lambda_1, \lambda_2, a, c, \delta)$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell) \frac{(a)_{k-1}}{(c)_{k-1}} |a_k|$$

$$+\sum_{k=1}^{\infty} \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell) \frac{(a)_{k-1}}{(c)_{k-1}} |b_k| \le 1.$$

3 Extreme points and distortion theorem

The following theorem investigates the extreme points of the convex hulls associated with the class

$\overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$ denoted by $clco\ \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$.

Theorem 3.1 *A function* $f_k \in clco\ \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$ *if and only if*

$$f_k(z) = \sum_{k=1}^{\infty} [X_k h_k(z) + Y_k g_k(z)], \tag{3.1}$$

where $h_1(z) = z$,

$$h_k(z) = z - \frac{2(1-\delta)}{[2k+\delta((-1)^k-1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k} z^k, (k \geq 2),$$

and

$$g_k(z) = z + \frac{2(1-\delta)}{[2k-\delta((-1)^k-1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k} \overline{z}^k, (k \geq 1),$$

$$X_k \geq 0, Y_k \geq 0, \quad \sum_{k=1}^{\infty} (X_k + Y_k) = 1.$$

Particularly, the extreme points of the class $\overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For the functions $f_k(z)$ of the form (3.1), we have

$$f_k(z) = z - \sum_{k=2}^{\infty} \frac{2(1-\delta)}{[2k+\delta((-1)^k-1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k} X_k z^k$$
$$+ \sum_{k=1}^{\infty} \frac{2(1-\delta)}{[2k-\delta((-1)^k-1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k} Y_k \overline{z}^k$$

Then, by using Theorem 2.3, we get

$$\sum_{k=2}^{\infty} \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k |a_k| + \sum_{k=1}^{\infty} \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k |b_k|$$

$$= \sum_{k=2}^{\infty} \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k \left(\frac{2(1-\delta)}{[2k+\delta((-1)^k-1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k} X_k\right)$$

$$+ \sum_{k=1}^{\infty} \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k \left(\frac{2(1-\delta)}{[2k-\delta((-1)^k-1)]\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k} Y_k\right)$$

$$= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1,$$

and so $f_k \in \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$.

Conversely, if $f_k \in clco\ \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$. Setting

$$X_k = \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k |a_k| \ (k \geq 2),$$

and

$$Y_k = \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k |b_k|,$$

we then obtain

$$f_k(z) = \sum_{k=1}^{\infty} [X_k h_k(z) + Y_k g_k(z)] \ asrequired.$$

The following theorem presents distortion bounds for functions f in the class $\overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$ which consequently provides a covering result for this class.

Theorem 3.2 *Let f be defined by (1.4). Then $f \in \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$. Then for $|z| = r < 1$, we have*

$$|f(z)| \leq (1 + |b_1|)r + \frac{1}{\left[\frac{\ell(1+\lambda_1+\lambda_2)+d}{\ell(1+\lambda_2)+d}\right]^m \Upsilon_2} \left\{\frac{1-\delta}{2} - \frac{1+\delta}{2}|b_1|\right\} r^2,$$

and

$$|f(z)| \geq (1-|b_1|)r - \frac{1}{\left[\frac{\ell(1+\lambda_1+\lambda_2)+d}{\ell(1+\lambda_2)+d}\right]^m \Upsilon_2}\left\{\frac{1-\delta}{2} - \frac{1+\delta}{2}|b_1|\right\} r^2.$$

The result is sharp.

Proof. We will prove the first inequality only, as the second follows a similar process and is not included here. Let $f \in \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$. Taking the absolute value of f we get the following

$$|f(z)| \leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \leq (1+|b_1|)r + r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|$$

$$\leq (1+|b_1|)r + \frac{(1-\delta)}{\left[\frac{\ell(1+\lambda_1+\lambda_2)+d}{\ell(1+\lambda_2)+d}\right]^m \Upsilon_2}\sum_{k=2}^{\infty}\frac{\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k}{(1-\delta)}(|a_k|+|b_k|)r^2$$

$$= (1+|b_1|)r + \frac{(1-\delta)r^2}{\left[\frac{\ell(1+\lambda_1+\lambda_2)+d}{\ell(1+\lambda_2)+d}\right]^m \Upsilon_2}\sum_{k=2}^{\infty}\left\{\frac{[2k+\delta((-1)^k-1)]}{4(1-\delta)}|a_k|\right.$$

$$\left. + \frac{[2k-\delta((-1)^k-1)]}{4(1-\delta)}|b_k|\right\}\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k$$

$$= (1+|b_1|)r + \frac{(1-\delta)r^2}{2\left[\frac{\ell(1+\lambda_1+\lambda_2)+d}{\ell(1+\lambda_2)+d}\right]^m \Upsilon_2}\sum_{k=2}^{\infty}\left\{\frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)}|a_k|\right.$$

$$\left. + \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)}|b_k|\right\}\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k$$

$$\leq (1+|b_1|)r + \frac{(1-\delta)r^2}{2\left[\frac{\ell(1+\lambda_1+\lambda_2)+d}{\ell(1+\lambda_2)+d}\right]^m \Upsilon_2}\left(1 - \frac{1+\delta}{1-\delta}|b_1|\right)$$

$$= (1+|b_1|)r + \frac{1}{\left[\frac{\ell(1+\lambda_1+\lambda_2)+d}{\ell(1+\lambda_2)+d}\right]^m \Upsilon_2}\left(\frac{1-\delta}{2} - \frac{1+\delta}{2}|b_1|\right) r^2.$$

The upper bound is sharp and the equality holds when

$$f(z) = z + \overline{b_1 z} + \frac{1}{\left[\frac{\ell(1+\lambda_1+\lambda_2)+d}{\ell(1+\lambda_2)+d}\right]^m \Upsilon_2}\left(\frac{1-\delta}{2} - \frac{1+\delta}{2}|b_1|\right)\overline{z}^2.$$

This concludes the proof of Theorem 3.2.

4 Convolution and convex combination

Before presenting the next theorem, it is necessary to define the convolution of two harmonic functions..

Let

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k \overline{z}^k,\ |b_1| < 1, \tag{4.1}$$

and

$$F(z) = z - \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k \overline{z}^k,\ (A_k \geq 0, B_k \geq 0). \tag{4.2}$$

Then the convolution of two harmonic functions f and F given by

$$(f * F)(z) = f(z) * F(z) = z - \sum_{k=2}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} b_k B_k \overline{z}^k. \tag{4.3}$$

Theorem 4.1 *Let* $f \in \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$ *and* $F \in \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \gamma)$ *for* $0 \le \gamma \le \delta < 1$. *Then*

$$f * F \in \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta) \subset \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \gamma).$$

Proof. The convolution $f * F$ is defined by (4.3). We want to show that the coefficient of $f * F$ satisfy the condition given by Theorem 2.3 .

For $F \in \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \gamma)$ we have

$$0 \le A_k \le 1, 0 \le B_k \le 1.$$

Now, for the convolution function $f * F$ we obtain

$$\begin{aligned} &\sum_{k=2}^{\infty} [2k + \delta((-1)^k - 1)]\Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell)\Upsilon_k |a_k| A_k \\ &+ \sum_{k=1}^{\infty} [2k - \delta((-1)^k - 1)]\Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell)\Upsilon_k |b_k| B_k \\ &\le \sum_{k=2}^{\infty} [2k + \delta((-1)^k - 1)]\Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell)\Upsilon_k |a_k| \\ &+ \sum_{k=1}^{\infty} [2k - \delta((-1)^k - 1)]\Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell)\Upsilon_k |b_k| \\ &\le 2(1-\delta) \le 2(1-\gamma). \end{aligned}$$

Hence, we have the desired result.

Let us now examine the properties of the convex combination of $\overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$.

Define the function $f_i(z)$, for $i = 1,2, \cdots$, as follows

$$f_i(z) = z - \sum_{k=2}^{\infty} |a_{k_i}| z^k + \sum_{k=1}^{\infty} |b_{k_i}| \overline{z}^k. \tag{4.4}$$

Theorem 4.2 *Let the functions* $f_i(z)$ *be defined by (4.4) be in the class*

$\overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$ for $i = 1,2, ...$. Then the functions $\xi_i(z)$ defined by

$$\xi_i(z) = \sum_{i=1}^{\infty} \tau_i f_i(z), \quad 0 \le \tau_i \le 1, \tag{4.5}$$

are also in the class $\overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$, where $\sum_{i=1}^{\infty} \tau_i = 1$.

Proof. Based on the given definition of $\xi_i(z)$, he convex combination of f_i can be expressed as follows

$$\xi_i(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} \tau_i |a_{k_i}|\right) z^k + \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} \tau_i |b_{k_i}|\right) \overline{z}^k.$$

Further, since $f_i(z) \in \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$ for $(i = 1,2, \cdots)$. Then by using Theorem 2.3, we have

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell)\Upsilon_k \left(\sum_{i=1}^{\infty} \tau_i |a_{k_i}|\right) \\ &+ \sum_{k=1}^{\infty} \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell)\Upsilon_k \left(\sum_{i=1}^{\infty} \tau_i |b_{k_i}|\right) \\ &= \sum_{i=1}^{\infty} \tau_i \left[\sum_{k=2}^{\infty} \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell)\Upsilon_k |a_{k_i}| \right. \\ &\left. + \sum_{k=1}^{\infty} \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)} \Lambda_d^{m,k}(\lambda_1, \lambda_2, \ell)\Upsilon_k |b_{k_i}|\right] \\ &\le \sum_{i=1}^{\infty} \tau_i = 1. \end{aligned}$$

This completes the proof of Theorem (4.2) .

5 Integral operator

Finally, our focus shifts to investigating the closure property of the class $\overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$ under the generalized Bernardi-Libera -Livingston integral operator $I_\mu(f)$ (see [15]), This operator is given by

$$I_\mu(f) = I_\mu(f(z)) = \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1} f(t)dt. \tag{5.1}$$

Theorem 5.1 *Let* $f(z) \in \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$. *Then* $I_\mu(f(z)) \in \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$.

Proof. Based on the representation of $I_\mu\big(f(z)\big)$, it can be concluded that

$$\begin{aligned} I_\mu\big(f(z)\big) &= \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1}h(t) + \overline{g(t)}dt \\ &= \frac{\mu+1}{z^\mu}\left(\int_0^z t^{\mu-1}\left(t - \sum_{k=2}^{\infty} a_k t^k\right)dt + \overline{\int_0^z t^{\mu-1}\left(\sum_{k=2}^{\infty} b_k t^k\right)dt}\right) \\ &= \frac{\mu+1}{z^\mu}\left(\int_0^z t^\mu dt - \sum_{k=2}^{\infty} a_k \int_0^z t^{\mu+k-1}dt + \sum_{k=2}^{\infty} b_k \int_0^z \overline{t^{\mu+k-1}dt}\right) \\ &= z - \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k \overline{z}^k, \end{aligned}$$

where $A_k = \frac{\mu+1}{\mu+k} a_k, B_k = \frac{\mu+1}{\mu+k} b_k$. Therefore

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)}\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k \frac{\mu+1}{\mu+k}|a_k| \\ &+ \sum_{k=1}^{\infty} \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)}\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k \frac{\mu+1}{\mu+k}|b_k| \\ &< \sum_{k=2}^{\infty} \frac{[2k+\delta((-1)^k-1)]}{2(1-\delta)}\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k|a_k| \\ &+ \sum_{k=1}^{\infty} \frac{[2k-\delta((-1)^k-1)]}{2(1-\delta)}\Lambda_d^{m,k}(\lambda_1,\lambda_2,\ell)\Upsilon_k|b_k| \le 1. \end{aligned}$$

Since $f \in \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$, therefore by Theorem 2.3 ,$I_\mu(f(z)) \in \overline{HS}_{S^*}^{m,q,s}(\lambda_1, \lambda_2, \ell, d, [\alpha_i, \beta_j], \delta)$.

Remark 1. Substituting $q = s+1, \alpha_i = 1(i = 1, \cdots, s+1)$,$\beta_j = 1(j = 1, \cdots, s)$, $\lambda_2 = d = 0$ and $\lambda_1 = \ell = 1$ into our findings, the results match those obtained by AL-Khal and Al-Kharsani [3].

Conclusion

In conclusion, the operator has enabled the creation of a new class of symmetric, complex-valued harmonic functions. This research defines a class of harmonic univalent functions using the operator $\mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m,q,s}[\alpha_i, \beta_j]$. presenting detailed findings on their coefficient bounds, distortion bounds, inclusion criteria, and closure properties under integral operators.

References

[1] F.M. Al-Oboudi, On univalent functions defined by derivative operator. International Journal of Mathematics and Mathematical Sciences, 27, 2004, 1429-1436.

[2] O. P. Ahuja and J. M. Jahangiri, Sakaguchi-type harmonic univalent functions, Sci. Math. Japonica., 59, 2004, no. 1, pp. 163-168.

[3] R. A. AL-Khal and H. A. Al-Kharsani, Salagean-type harmonic univalent functions with respect to symmetric points, Australian J. Math. Analysis Appl., 4, 2007, no. 1, Art. 6, pp.1-13.

[4] A. Cătas, On certain classes of p-valent functions defined by new multiplier transformations, TC IstandulKultur University Publications, Proceedings of the International Symposim on Geometric Function Theory and Applications (GFTA'07), Istanbul, Turkey. August 2007, Vol 91,2008 , pp. 241-250.

[5] J. Clunie and T. Sheill-Small, Harmonic univalent functions, Ann. Acad. Aci. Fenn. Ser. A.I. Math., 9 , 1984, pp, 3-25.

[6] N.E. Cho and H.M. Srivastava, Argument estimates of certain analytic functions defined by a class of multiplier transformations, Math. Comput. Modelling, 37(1-2), 2003, pp. 39-49.

[7] R. N. Das and P. Singh, On subclasses of schlicht mapping, Indian J. Pure Appl., Math. 8, 1977, pp. 864-872.

[8] J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric functiction, Integral Transforms and Special Functions, vol. 14, no. 1, 2003, pp. 7-18.

[9] R. M. El-Ashwah and M. K. Aouf, A. Shamandy and S. M. El-Deeb, Certain class of harmonic starlike functions with respect to symmetric points defined by linear operator, Acta Universitatis Apulensis, No. 34/2013 pp. 207-222, ISSN: 1582-5329.

[10] E.El-Yaguby and M.Darus, Subclasses of analytic functions defined by new generalized derivative operator, Journal of Quality Measurement and Analysis, vol. 9, no.1, 2013, pp. 47-56.

[11] E.A.Eljamal and M.Darus, Subordination results defined by a new differential operator, Acta Universitatis Apulensis, vol. 27, 2011, pp.121-126.

[12] M.Darus and K.Al-Shaqsi, Differential sandwich theorems with generalize derivative operator. International Journal of Computing and Mathematical Sciences, vol. 22, 2008, pp.75-78.

[13] J.M. Jahangiri, Harmonic functions starlike in the unit disc., J. Math. Anal. Appl., vol. 235,1999, pp. 470- 477.

[14] J.M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya, Starlikeness of Rucheweyh type harmonic univalent functions, J. Indian Acad. Math., vol. 26, 2004 , pp. 191-200.

[15] R. J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc.,vol.16, 1956 , pp. 755 - 758.

[16] G. Murugusundaramoorthy, K. Vijaya and M. K. Aouf, A class of harmonic starlike functions with respect to other points defined by Dziok-Srivastava operator, J. Math. Appl., 2008 , no. 30, pp. 113-124.

[17] A.Oshah and M.Darus, Differential sandwich theorems with a new generalized derivative operator , Advances in Mathematics: Scientific Journal, vol. 3 , no.2, 2014, pp. 117-124 ISSN 1857-8365.

[18] G.S. Sălăgean, Subclasses of univalent functions, in Proceedings of the Complex Analysis 5th Romanian Finnish Seminar Part ,11013 ,1983 , pp.362-372.

[19] K. Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan,vol 11, 1959, pp.72-75.

[20] S.R. Swamy, Inclusion properties of certain subclasses of anlytic functions, International Mathematical Forum, vol.7, no. 36 , 2012, pp. 1751-1760.

[21] B.A. Uralegaddi and C. Somanatha, Certain classes of univalent functions, In Current Topics in Analytic Function Theory, River Edge, NJ:World Scientific, 1992, pp. 371-374.

[22] Al Amoush, A. G., & Darus, M. Harmonic starlike functions with respect to symmetric points. *Matematika*,vol 32, no.2, 2017, pp.121-131.

[23] K. Deepali & K. Raj & Y. Sibel. (2020). A Class of Harmonic Starlike Functions defined by Multiplier Transformation. Advances in Mathematics: Scientific Journal, vol.9 2020, no.1, pp.455–469