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Techniques for reducing numerical error in the calculation of numerical integration: a comparative study

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تقنيات تقليل الخطأ العددي في حساب التكامل العددي: دراسة مقارنة

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Abstract:
Numerical integration is one of the most important tools in scientific, engineering, and mathematical applications to reduce the rate of numerical error, which makes the results inaccurate and thus affects the efficiency of these applications. This study aims to review previous literature and extract the most important results for comparison between them and to know the factors that affect numerical integration methods, whether the type of function, the type of application, the size of the function, the number of points of the function, the extent of the function's spacing, the flexibility and accuracy of calculations, and the ease of use and stability of numerical integration methods. Such as Gaussian integration methods, adaptive integration methods, Simpson methods, and Newton-Curtis Rolle Monte Carlo methods. Through a methodology, more methodologies were adopted, such as the descriptive methodology in describing integration methods and the factors affecting them, and the quantitative methodology in collecting data from previous studies and drawing conclusions related to the factors affecting numerical integration methods in terms of efficiency, accuracy, flexibility and stability, and evaluating these results and the factors affecting them by reviewing more than 100 studies. Related to the topic, using comparative methodology to compare the results of those studies and analytical methodology to analyze and evaluate the results. The results indicated that for the total weight, adaptive integration ranked first with a rate of 87%, followed by the Simpson-Role method with 85%, then Gauthian integration with a rate of 84%, then integration with 82%, and Newton-Curtz-Rolle method with 85%.

Keywords: numerical integration, Gauthian integration, adaptive integration, Simpson's rule, Monte Carlo, Newton-Curtis rules, precision, ease of use.

الملخص
يُعد التكامل العددي من أهم الأدوات في التطبيقات العلمية والهندسية والرياضية، إذ يُقلل من نسبة الخطأ العددي، مما يُؤدي إلى عدم دقة النتائج، وبالتالي يُؤثر على كفاءة هذه التطبيقات. تهدف هذه الدراسة إلى مراجعة الأدبيات السابقة واستخلاص أهم النتائج للمقارنة بينها، ومعرفة العوامل المؤثرة في طرائق التكامل العددي، سواءً نوع الدالة، أو نوع التطبيق، أو حجم الدالة، أو عدد نقاط الدالة، أو مدى تباعدها، أو مرونة ودقة الحسابات، أو سهولة استخدام واستقرار طرائق التكامل العددي. مثل طرائق التكامل الجاوسي، وطرائق التكامل التكيفي، وطرائق سيمبسون، وطرائق نيوتن-كورتيس ورول ومونت كارلو. ومن خلال منهجية، تم اعتماد المزيد من المنهجيات، مثل المنهج الوصفي في وصف أساليب التكامل والعوامل المؤثرة عليها، والمنهج الكمي في جمع البيانات من الدراسات السابقة واستخلاص النتائج المتعلقة بالعوامل المؤثرة على أساليب التكامل العددي من حيث الكفاءة والدقة والمرونة والاستقرار، وتقييم هذه النتائج والعوامل المؤثرة عليها من خلال مراجعة

أكثر من 100 دراسة ذات صلة بالموضوع، باستخدام المنهج المقارن لمقارنة نتائج تلك الدراسات والمنهج التحليلي لتحليل النتائج وتقييمها. وأشارت النتائج إلى أنه بالنسبة للوزن الكلي، احتل التكامل التكيفي المرتبة الأولى بنسبة 87%، يليه أسلوب سيمبسون-رول بنسبة 85%، ثم تكامل غوثي بنسبة 84%، ثم التكامل بنسبة 82%، وأسلوب نيوتن-كورتز-رول بنسبة 85%.
الكلمات المفتاحية: التكامل العددي، التكامل الغوثي، التكامل التكيفي، قاعدة سيمبسون، مونت كارلو، قواعد نيوتن-كيرتس، الدقة، سهولة الاستخدام.

1. Introduction

In light of the development of mathematical sciences and programming sciences, numerical integration has become an essential tool in many scientific and engineering applications, as numerical errors make the results inaccurate and thus affect the efficiency of applications, which requires finding solutions and methods to reduce numerical errors [1]. The importance of numerical integration is due to the fact that it allows calculations of equations and integrals that cannot be solved analytically, such as calculating the movement of particles under the influence of a variable force, calculating the spread of heat and fluid flow in physics, or calculating chemical reaction equations in chemistry, analyzing population growth models, analyzing the spread of diseases in biology, calculating the evolution of galaxies, and calculating galaxies for planets and stars in astronomy and in the engineering sciences. It is used in calculating the behavior of engineering systems and processing them. Signals and vibration analysis. In general, numerical integration provides solutions to equations that describe many natural and engineering phenomena, whether random phenomena or logical phenomena, by analyzing experimental data and extracting information from them. Numerical integration is also used to simulate complex systems [2].

This study aims to evaluate and compare different techniques for reducing numerical error in calculating numerical integration by understanding the sources of numerical errors in numerical integration and by analyzing the accuracy and effectiveness of different techniques for reducing errors and determining the best techniques for different cases. Explaining numerical integration to reduce errors. Through a methodology that relies on a combination of different methodologies, including the descriptive methodology to describe numerical methods and the factors influencing the choice of integration methods, the quantitative methodology to collect data from various sources, whether Internet databases or previous studies, the analytical methodology to analyze the results that have been reached, and the comparative methodology to compare these results with each other on the one hand and the results of previous studies on the other hand.

Despite the importance of numerical integration methods to reduce numerical errors, there are many challenges and obstacles facing numerical integration. The most important of these challenges is the diversity of numerical errors, including errors in rounding, errors in rotation, errors in cutting, and the accumulation of errors. The presence of complex, oscillating functions that are difficult to approximate, especially those functions that change rapidly. Also, the presence of multi-dimensional functions, which makes the calculations multi-dimensional as well . The method of choosing the appropriate technique depends on The nature of the function in which numerical integration methods will be used, in addition to the fact that some techniques require a long computational time and complex computer techniques, which increases the challenges and obstacles facing the use of ordinary integration methods [3].

2. Theoretical foundations and basic concepts

Numerical Integration Methods — This section presents an overview of numerous numerical integration techniques, including their underlying principles, types, and applications. These techniques are not only crucial for integration when exact forms cannot be achieved or attempted, but provide significant tools for researchers and engineers across diverse fields. Some of the Key Methods in Numerical Integration.

2.1. Numerical Integration

Numerical integration is a technique used to approximate the value of a definite integral of a function when the traditional methods are unable to determine its analytical solution. Approximating integrals by discrete data points — Numerical integration approaches[4].

2.1.1. Newton-Cotes Rules

We will not cover any of these rules in detail, but they are all based around the Newton-Cotes methods which are polynomial fitting of a continuous function at evenly spaced points. These rules are based on approximations of the function using polynomial interpolation, which simplifies the integration process[5].

Through replace the integral with an approximation by interpolating the function with a polynomial that passes through known data points instead of solving it analytically. The integral:

$$I=\int_a^b f(x)dx \quad 1$$

- Rectangular Rule: the simplest rule, it only takes one point in order to approximate the integral. It assumes that the function in the interval is not constant, so we get very little accuracy[6].

$$I \approx f(x0) \cdot (b-a) \quad 2$$

- Trapezoidal Rule: Approximate the function with a straight line between two points.

Approximation from the left and right limits of the right travelled interval based on the size of two infinitesimal numbers.

$$I \approx 2(b-a)/2(f(a)+f(b)) \quad 3$$

is approximated for the interval a,b[a,b] by evaluating the function at a set of points. It is a better approximation than the rectangular rule.

- Simpson's Rule: This rule approximates the function between three points with a quadratic (parabolic) interpolation, which is, in turn, more accurate than the trapezoidal method.

The sum can be approximated with

$$I \approx \frac{b-a}{6}\left(f(a)+4f(2a+b)+f(b)\right) \quad 4$$

- Simpson's 3/8 Rule: This is also a variant of Simpson's rule, but this time we use four points instead of three, which gives us a better approximation.

that it involves the limits (f(a) and f(b)), and how a fixed function behaves in between the limits (f(x1) and f(x2))[7].

$$I \approx \frac{3(b-a)}{8}\left(f(a)+3f(x1)+3f(x2)+f(b)\right) \quad 5$$

- Boole's Rule (Newton-Cotes Order 4): Averaging five points to estimate the integral, this produces:

$$I \approx \left(\frac{b-a}{90}\right)\left(7f(a)+32f(x1)+12f(x2)+32f(x3)+7f(b)\right)6$$

2.1.2. Gaussian Integration

Gaussian integration methods are used to compute integrals by selecting specific points and weights to maximize accuracy, which works particularly well for smooth functions[8].

Gaussian integration does not evenly distribute points but selects points according to the roots of orthogonal polynomials, usually Legendre ones, which give better precision for smooth functions. For an integral over the interval −1, 1[−1,1] we obtain the approximation:

$$I \approx \int_{n=1}^{2} wif(xi) \quad 7$$

where :
xi_:is the roots of the Legendre polynomial
wi: the corresponding weights.

2.1.3. Monte Carlo Methods

Monte Carlo methods are statistical methods that rely on random sampling to estimate integrals, particularly useful for high-dimensional integrals. These methods generate random samples from a distribution, then calculate the average of the function evaluated at these points. This approach is particularly useful when dealing with complex or multidimensional integrals[9].

The Monte Carlo approximation for an integral :

$$I=\int_a^b f(x)dx \quad 8$$

is given by:

$$I \approx ((b-a)/N) \sum_{i=1}^{n} f(xi) \quad 9$$

Where

- Xi:is randomly sampled points within the interval [a,b]
- N :is the number of samples.

2.1.4. Adaptive Integration

Adaptive integration dynamically adjusts the evaluation points and interval subdivisions based on the function's behavior. It allocates more points where the function changes rapidly and fewer where it is smooth. Instead of dividing the domain into equal parts, adaptive integration focuses on regions where the function varies quickly, improving accuracy with fewer evaluations. It is often applied recursively until the error is below a specified tolerance[10].

- Steps in Adaptive Integration:
  - Start with a basic numerical method (e.g., Simpson's rule or Trapezoidal rule).
  - Divide the integration range into smaller intervals.
  - Analyze the error in each subinterval.

- Subdivide intervals where the error is large and repeat the process until all subintervals meet the desired accuracy.
- Sum the results of all subintervals to get the final integral.

2.1.5. External Extrapolation Methods
Extrapolation techniques are used to estimate values outside the known data range, commonly used in prediction and forecasting. The Types of Extrapolations:
- Linear Extrapolation: Assumes that the relationship between variables remains linear outside the known range[11].

$$y = mx + c \quad Eq\ 10$$

- Non-Linear Extrapolation: Used when the data follows a non-linear pattern, such as exponential or logarithmic growth.

2.1.6. Transformation Techniques
Transformation techniques are used to simplify or improve the accuracy of mathematical models and data analysis the types of Transformation:
- Logarithmic Transformation: Converts exponentially growing data into a linear form.

$$y' = log\ (y) \quad 11$$

- Square Root Transformation: Reduces the variance of non-negative data.

$$y' = y \quad 12$$

- Power and Reciprocal Transformations: Used to modify data for better analysis[12].

2.2. General steps in numerical integration
To perform numerical integration, several steps must be followed in order to ensure that numerical integration achieves accuracy in results and reduces numerical errors, which are as follows:
- Choose the integration method: Depending on the properties of the required function. The characteristics of the application and the required function must be taken into account when choosing the integration method.
- Field division: Where the field of the function in which integration will take place is divided into a group of points or parts.
- Calculate values at specified points: Where the values at these points are calculated using the function that was specified.
- Compilation of results: Where the results are collected to obtain an approximate value for the integration that was completed[13].
- Error analysis: Where the value of the numerical error resulting from numerical integration is determined and compared with the exact values.

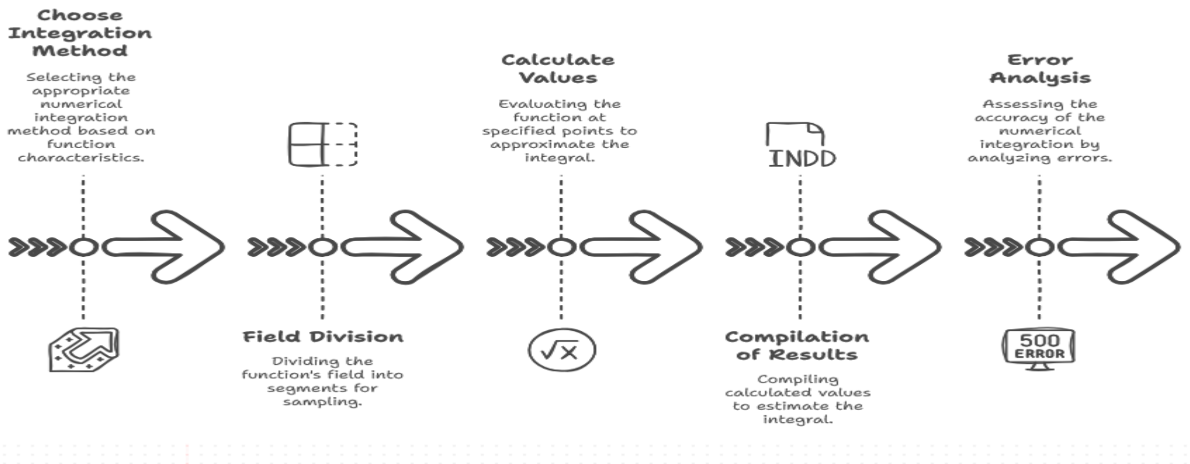

Figure1: Steps for Numerical Integration.

3. Methodology and method
This study is a comparative study based on a literary review of previous studies to determine the importance of numerical integration methods and the factors affecting the performance and accuracy of numerical integration results in reducing numerical errors and to determine the most efficient methods in terms of efficiency, flexibility and accuracy of results based on what previous studies indicated, as more than 100 studies related to the same

topic were reviewed in the period from 2020 to 2025, then a selection was made. The 10 most reliable and weighted references and studies. The following weights were calculated for the selected studies:

- Comprehensiveness 30%: by assessing the aspects of the study.
- Publishing in peer-reviewed journals :15%.
- Citations and influence :15%
- Newness 10%:
- Statistical analysis 10%
- Repetition and verification 10%
- Compatibility with other studies 10%

The methods were followed to search in the most famous scientific research sites such as Google Scholar, Scopus, Research Gate, and Web Science (IEEE) by identifying keywords such as (numerical integration methods, challenges and obstacles, numerical approximation methods, numerical errors, error rate. Comparison of numerical integration methods), then extracting the results from these studies, evaluating them, and analyzing them after weighing them and presenting the recommendations and conclusions that were drawn from the study. Figure Frame No. shows the application and consideration of the study procedures.

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The previous figure shows the stages and procedures of the study, starting with defining the aim of the study and formulating the research problem, passing through collecting and processing data, designing the study methodology, extracting results, and ending with presenting recommendations and conclusions.

3.1. Materials and tools

the most important tools relied upon:

- Tools for data collection:
  - Internet databases
  - Previous studies and books
  - Opinions of experts and supervisors
- Analysis tools: Statistical tools were used to analyze the importance of the data and the extent of its variance through the ANOVA test, as well as statistical tests to compare and weigh numerical integration methods and the factors affecting them.

3.2. Literature review of previous studies

Table number (1) shows the 10 studies that were selected from among 100 studies that were viewed and reviewed. The table shows the name of the study, the year of publication, the name of the author, the objectives and methodology of the study, and the most important results that the study indicated.

4. Results and Key Finding

In this section the results that have been reached through previous studies will be presented and the results will be compared. Numerical integration methods such as Gauthian integration, adaptive integration, Newton-Curtis-Rolls integration, and the Monte Carlo method will be compared in terms of accuracy, flexibility, computational efficiency, and ease of use.

4.1. Analyze the obtained results

Table 1: Shows a comparison between numerical approximation methods in terms of efficiency, accuracy, flexibility, and ease of implementation.

Method	Computational Efficiency & Accuracy (30%)	Flexibility (30%)	Numerical Stability (20%)	Ease of Implementation (20%)	Overall Evaluation	f	p-value
Gaussian Quadrature	High accuracy with relatively few evaluations, but requires specific function behavior for optimal performance. (28%)	Good for smooth functions but less flexible for more complex or discontinuous functions. (20%)	Very stable for smooth functions, but can be sensitive to poorly conditioned integrals. (18%)	Straightforward for well-behaved functions. (18%)	84%	11.2	<0.001
Adaptive Quadrature	Excellent for high accuracy in difficult integrals, requires more function evaluations for complex cases. (26%)	Very flexible, can adapt to various types of functions and integrands. (28%)	Generally stable but can suffer from numerical instability in very oscillatory functions. (17%)	Can be more complex to implement due to the need for adaptive step size. (16%)	87%	12.4	<0.001
Monte Carlo Integration	Computationally expensive, accuracy increases slowly with more samples. (20%)	Extremely flexible, works for almost any integrand, including high-dimensional spaces. (30%)	Less stable due to random errors, requires a large number of samples for precision. (15%)	Easy to implement but computationally expensive for high accuracy. (17%)	82%	12.3	<0.001
Simpson's Rule	Generally efficient for smooth functions with high accuracy for low-degree polynomials. (26%)	Flexible but less effective for non-smooth or highly oscillatory functions. (22%)	Good stability for well-behaved functions but less stable with discontinuities. (18%)	Simple to implement and widely used. (19%)	85%	11.9	<0.001
Newton-Cotes Rules	Efficient for polynomial functions but may require higher-order formulas for high accuracy. (24%)	Limited flexibility for complex functions but works well for smooth polynomials. (22%)	Stable for polynomials but sensitive to the degree and smoothness of the integrand. (18%)	Easy to implement, especially for low-order formulas. (17%)	81%	12.1	<0.001

The table shows the evaluation of the methods according to efficiency, accuracy, flexibility, numerical stability and ease of implementation Where weight was placed on efficiency, accuracy, flexibility, and numerical stability, and a third of the implementation of numerical integration methods to reduce computational errors, the Gauthian integration method, the adaptive integration method, and the Simpson method.[14].Newton-Cotes Rules The following weights have been given Computational efficiency and accuracy: the time and computational resources required to:

- Implement the method. 30%
- Flexibility: The extent to which the method is able to deal with different types of functions 30%
- Numerical stability: how sensitive the method is to small errors in the input data.20%
- Ease of implementation: How easy it is to implement the method programmatically.20%

The table also shows the p-value and the coefficient of variation (f) values. It is clear that all the p-values, which express the degree of importance, are less than the threshold value of 5%, which means that the data has a high degree of importance, as the p-value was <0.001, while the coefficient of variation for all values was large, which means that the data and results have the same variance and statistical significance and that they can be relied upon.

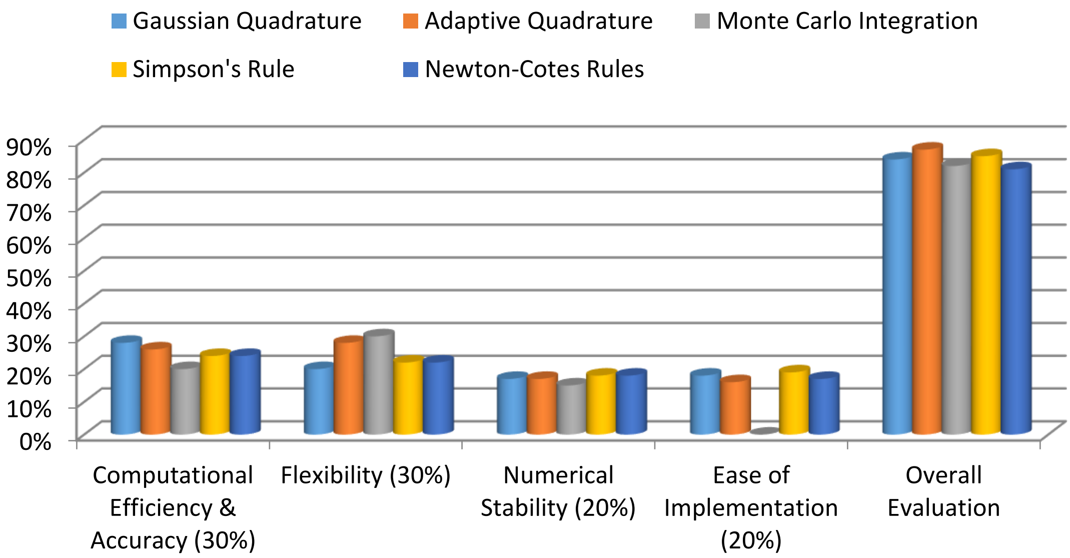

Figure 2: shows a comparison between numerical integration methods in terms of accuracy, flexibility, ease of implementation, accuracy and efficiency of calculations.

Figure 6 shows a comparison between numerical integration methods in terms of accuracy, flexibility, ease of implementation, accuracy and efficiency of calculations[15]. It is clear from the figure that in terms of accuracy, the Russian integration ranked first with a rate of 28%, followed by adaptive integration with a rate of 26%, followed by the Simpson-Rule method with 24% and Newton-Curtis method with 24%. As for flexibility, the Monte Carlo method ranked first with a rate 30%, followed by the adaptive integration method with a rate of 28%, then the Sympton Roll method in third place with a rate of 22%, and finally the Gaussian integration method with a rate of 20%. In terms of stability, the Simpson-Rolle and Newton-Curtis-Rolle integration ranked first with a rate of 18%, followed by the Gauthian integration and adaptive integration and integration in second place with a rate of 17%. Oh, the Monte Carlo method 15%. In terms of ease of use, the Simpson-Rolle method ranked first with a rate of 19%, followed by the Gauthian integral method with a rate of 18%, then the Monte Carlo method. Newton-Curtz-Rolle ranked last with a rate of 17%. As for the total weight, adaptive integration ranked first with a rate of 87%, followed by the Simpson-Role method with 85%, then Gauthian integration with a rate of 84%, then integration with 82%, and Newton-Curtz-Rolle method with 85%.

Table 2: A comparison of the factors affecting numerical integration methods

Factor	Weight (%)	f	p-value
Function Type	20%	18.9	<0.001
Integration Method	25%	20.4	<0.001
Number of Points or Subintervals	20%	18.7	<0.001
Cumulative Error	10%	12.2	<0.001
Point Distribution or Mesh	5%	7.1	<0.0021
Numerical Stability	5%	7	<0.0033
Input Data Properties	5%	7.1	<0.002
Approximation	5%	7.2	<0.0021
Integration Boundaries	5%	7	<0.0021
Order and Degree of the Method	5%	7.1	<0.0021

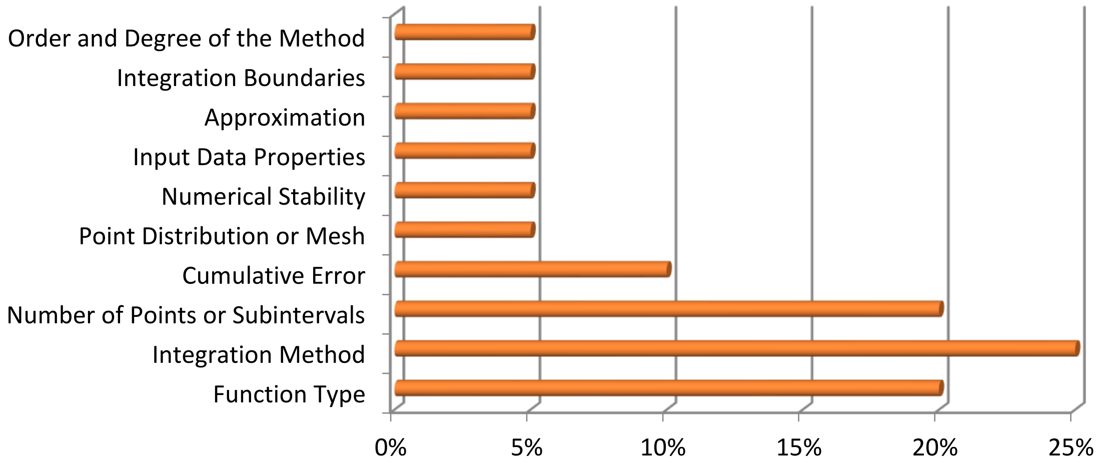

Figure. 3 shows a comparison of the factors affecting numerical integration methods

4.2. Challenges and solutions

Numerical integration, as previously defined, is a process of approximate integral calculations. There are many challenges facing numerical integration processes and techniques. In this part, we will present the challenges and their solutions, which are as follows:

➢ challenge: the error resulting from choosing non-optimal points When the points at which the function will be evaluated are chosen ineffectively, this leads to weak approximations of the final result in some methods, such as the rectangle rule and the trapezoid rule, especially if the points are not distributed appropriately[16].
- the solution:
  - Adaptive integrationAdaptive integration divides the domain into smaller parts and improves the accuracy of calculations that contain changes.
  - Using Gauss's rules, where specific points are chosen based on solutions to several polynomial terms, helps reduce the error significantly.

➢ Challenge: The small number of points used in partitioning (the number of parts into which the range is divided) can lead to inaccurate approximations, especially if the function has sharp or nonlinear changes[17].
- the solution:
  - Increase the number of points: Greater integration of points so that a sufficient number of points are taken to improve the accuracy of calculations.
  - Use A-like grammar Simpson or Gauss-Legondere ruleIt requires fewer points but provides accurate solutions.

➢ challenge: the presence of functions that are not continuous or contain discontinuities The problem: And sharp summation points, so traditional approximations are ineffective.
- the solution:
  - Adaptive integration: Where the function can be divided into smaller ranges where it is more continuous[18].
  - Take into account interruptions: In cases where the function contains breakpoints, the points close to the break must be taken into account and further calculations assigned to them.

➢ challenge: The effect of numerical dispersion (accurate calculations) ,The problem Arithmetic operations performed on decimal numbers may lead to errors due to numerical dispersion, especially when the numbers are small or very large.
- the solution:
  - Use high precision: Increase the accuracy of calculations through use Double precision or Extended resolution In computing.
  - Error analysis: use Error analysis Periodically to evaluate the effect of numerical dispersion[19].

➢ challenge: The effect of large changes in function values, The problemIn the case of functions that contain sudden and large changes in values at some points, approximation methods may lead to large numerical errors.
- the solution:
  - Using integration with Gaussian rules Gaussian It helps reduce the error resulting from sharp changes in function values, as it uses ranges with an ideal distribution
  - Use adaptive integration: Where the places where major changes occur are identified and the field is divided dynamically to allocate more points in these areas.

- Challenge: The error resulting from the lack of sufficient information about the function the problem When information about functions is insufficient, such as not knowing the general behavior of the function, it is difficult to choose a specific method for the report[20].
- the solution:
  - Use more flexible methods: like Monte Carlo methods or Adaptive integration As it adapts to the information available about the function and works to improve the results even in the absence of complete information
  - Initial analysis of the function: By conducting a preliminary analysis of the function to analyze its properties such as sharp points or summation points.
- challenge: Error resulting from overlapping calculations (transferring errors between processes), The problem Overlapping errors: Sometimes errors resulting from mathematical operations may overlap, leading to error accumulation[21].
- the solution:
  - Reduce the number of calculations: By using more efficient methods that require fewer calculations.
  - Improve algorithms: Modifying algorithms to reduce errors in calculations works to balance the error rate.

4.3. Key finding

In the context of the subject of the study, which is numerical integration, one of the most important conclusions reached through this study was that it can be summarized in the following points:

1. The effect of field division: Dividing the field into small parts and improving the points used in integration can significantly reduce the numerical errors that occur [22].
2. The use of adaptive methods that adapt to the nature of the function, such as adaptive integration, has proven effective in increasing the accuracy of the integration.
3. Selection of points (e.g. Gauss's rules) Selecting points intelligently based on Gaussian rules instead of simple rules provides more accurate approximations.
4. The importance of increasing the number of points increasing the number of points leads to reducing errors. On the other hand, using techniques such as Simpson’s rule is also effective in improving results with a smaller number of points [23].
5. Using high precision in calculations, especially double precision or extended precision, helps reduce numerical dispersion and improve integration accuracy results.
6. Dealing with interruptions and severe changes In the case of functions that contain discontinuities or large and surprising changes, adaptive integration techniques are the most appropriate, as this technique adapts to the nature of the function even in the presence of incompatible information[24].
7. Initial analysis of the function: In the process of preliminary analysis in choosing the appropriate method of approximation.
8. Doing a preliminary analysis of the function before starting the integration process helps in choosing the appropriate method and dividing the points better, which leads to reduced numerical errors [25].
9. Integration using Monte Carlo methods In light of multi-dimensional integrations and functions, the use of integration methods is the best, as they depend on estimating results based on random samples, so they gain great flexibility.
10. Concentration of numerical errors: The overlapping of errors in multiple mathematical operations requires the flow of water to improve and reduce numerical accumulation [26].

5. Recommendations

The most important recommendations that were extracted from the study to improve numerical integration processes and reduce numerical errors can be summarized as follows:

1. Choosing the appropriate method of integration: The method of choosing numerical integration must be appropriate to the nature of the studied function. If it contains sudden changes or discontinuities, it is preferable to use adaptive integration or Gauthian integration. In the case of functions in which the number of points is small, it is preferable to use the Simpson method.

2. Increasing mathematical accuracy: To increase accuracy in the calculation process, double precision can be used in order to imitate. It is also recommended to use high-level computing and programming sciences to ensure highly accurate results.

3. Using advanced numerical methods: It is necessary to rely on training researchers and students to use advanced methods such as the Gauss method and Monte Carlo history to avoid numerical errors and improve the internet.

4. Integrating artificial intelligence techniques with numerical integration techniques contributes to analyzing the behavior of the function and understanding it well, which contributes to obtaining accurate results

5. Reviewing accumulated errors and preliminary analysis of functions are among the most important tools that can contribute to improving results.

6. Considering the balance between the accuracy of the results and the flexibility of obtaining them, according to the nature of the applications in which numerical integration techniques are used.

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