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On the Connectedness of Graphic Topology

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حول ترابط الطوبولوجيا البيانية

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Abstract:
In this study, we delve into the intricate aspects of graphic topology. We establish the special cases in which topological spaces have a graphic topology, and relation between the graphic topology and certain graphs, we present connected graphic topology, moreover, show that the graphic topological space is irreducible if and only if the graph is empty. Finally, we offer the of connectivity of graph through special conditions on adjacency sets.

Keywords: graph theory, graphic topology, connectedness, irreducible set.

الملخص
في هذه الورقة، تم تقديم عرض تفصيلي للفضاء التبولوجي البياني وعلاقته ببعض الأنواع الخاصة من البيان. تم تحليل خاصية الترابط في هذا الفضاء، مع التركيز على الشروط التي تؤثر فيها. علاوة على ذلك تم التوصل إلى أنَ الفضاء البياني يكون غير قابل للاختزال إذا وفقط كان البيان خالي. وأخيراً تمت مناقشة إمكانية ترابط البيان من خلال تقديم شروط خاصة على المجموعات المجاورة في الفضاء.

الكلمات المفتاحية: نظرية البيان، التبولوجي البياني، الترابط، المجموعات غير القابلة للاختزال.

1. Introduction

Graph theory stands as one of the most fundamental and significant branches of mathematics and interesting branches of mathematics nowadays, which has many applications in our real life, it is a fundamental and powerful analytics tool for many domains, that not only deepens our understanding of various mathematical concepts but also helps us to optimally solve many practical problems which illustrates its widespread application. Defining endowing a graph with a topology enhances its structural depth and complexity., from this point of view, many topologies have been defined with different bases by lots of researchers, a number of them studied generating Topological structures defined on the vertex set of directed graphs provide deeper insights into their properties and relationships, others on undirected graphs (see [1,2,4,8,10,11,14]). Research continues in graphical spaces as a fertile and evolving field, and in recent years there has been great progress in the study of graphic topological spaces, where researchers have built new topologies different from their predecessors and have been employed in more precise fields such as biomathematics and cardiology. One of these topologies was presented by Jafarian and others in [13] which defined on the vertices set of a locally finite undirected graph without isolated vertex and was called the graphic topology. Later many research papers dealt with this topology in more detail, they can be found in [3,7,8]. The connectivity property is a topic of interest in graphic topological spaces under study. Zomam et al. (2021) explored conditions for Alexandroff spaces to be graphic topological spaces, addressing an open problem in the field. Zomam & Dammak (2022) introduced the Z-graphic topology, which preserves graph connectivity and is homeomorphic for isomorphic graphs. They provided a topology is Z-graphic if and only if it satisfies both the necessary and sufficient conditions defining its graphical structure. In related work, Hu (2010) and Figueroa & Rivera-Campo (2008) investigated the connectivity of tree graphs defined by cycle sets. Both studies presented necessary and sufficient conditions for the tree graph to be connected, with Figueroa & Rivera-Campo (2008) focusing on cases where each edge belongs to at most two cycles. These papers collectively contribute to understanding graph connectivity in various topological contexts. In this paper, our motivation is to give an elementary base for studying the fundamental properties of the connectivity of graphic topological spaces and suggest the conditions affecting the connectedness

The paper has four sections, In section 2, we give some fundamental definitions of a graph, topological space, and graphic topology. Section 3 is dedicated to some introductory properties and distinctive results of graphic topology, the last section is devoted to connected graphic topology, Furthermore, the sufficient conditions for the connectivity of graphic topology are introduced.

2. Preliminaries

The following section introduces the basic concepts of graph theory. [5,12,17], and topological spaces [9], which are utilized in this study. Additionally, the basic concept of graphic topology is introduced.

A graph G is defined as a pair of sets (V, E), where V A non-empty set whose elements are referred to as vertices, along with a set E consisting of unordered pairs known as edges, forms the structure of a graph. each edge connects two vertices, u and v, which are identified as its end vertices $e = (\mathrm{u}, \mathrm{v})$ referred to as adjacent. Additionally, each vertex is said to be incident with an edge if it is one of the edge's endpoints e if $\mathrm{v} \in e$. edges that share the same end vertices are known as parallel edges, while an edge of the form (v, v) is referred to as a loop A graph is considered simple if it does not contain parallel edges or loops. A graph with no edges is called empty, while a graph with no vertices is referred to as a null graph. Additionally, a graph consisting of a single vertex is known as trivial. A set of vertices in a graph that are mutually non-adjacent is termed an independent set. The degree of a vertex v, denoted as $d(\mathrm{v})$, represents the number of edges incident to it.. $d(\mathrm{v})$, The number of edges incident to a vertex defines its degree. A vertex with a degree of zero is known as an isolated vertex, while a vertex with a degree of one is called a pendant vertex. A graph is classified as finite if both V and E are finite sets; otherwise, it is considered infinite. An infinite graph is termed locally finite if every vertex within it has a finite degree.

A subgraph of a graph G consists of vertices that belong to V and edges that are contained within E. A graph is considered connected if any vertex can be reached from any other vertex by following the edges, and disconnected otherwise. A component of graph G is a connected subgraph that is not part of a larger connected subgraph.

Topology is a branch of mathematics concerned with the study of space and its properties under continuous deformation. A topology τ\tau on a set X is defined as a collection of subsets of X, referred to as open sets, such that the intersection of any two subsets τ is a member of τ, the union of any subsets of τ is a member of τ, also the empty set and the whole set are members of τ. The pair (X, τ). A set X equipped with a topology is referred to as a topological space, or simply called a topological space. The discrete topology on X is the topology that includes all possible subsets of X, while the topology that consists only of X and the empty set is known as the indiscrete or trivial topology. A collection $\mathscr{B} \subseteq \tau$ is called a basis of (X, τ) if every nonempty open set is the union of certain members of $\mathscr{B}$, and the collection $\mathcal{S} \subseteq \tau$ is called a sub-basis of (X, τ) if the family of finite intersections of members of $\mathcal{S}$ is a basis of (X, τ). A space X is considered connected if it cannot be expressed as the disjoint union of two non-empty open subsets. Equivalently, X is connected if its only subsets that are both open and closed are X itself and the empty set. X are $\emptyset$ and X. A component C of a topological space X is a maximal connected subset of X; that is, C is connected and it is not a proper subset of any connected subset of X

In what follows, let $G = (V, E)$ A simple, undirected, locally finite graph without isolated vertices is one in which every vertex has a finite degree and no edges are repeated or self-looping. The graphic topology τ_G on the set V is the topology generated by the sub-basis $\mathcal{S}_G$ such that $\mathcal{S}_G = \{A_x : x \in V\}$, whenever A_x is the set of all vertices adjacent to x, the pair (V, τ_G) is called graphic topological space.

All graphs throughout this paper are locally finite simple graph.

3. Graphic topology

Graphic topology in mathematics, a topological graph represents a graph within a plane, where its vertices correspond to distinct points and its edges define connections between them. define the adjacency sets that generate the topology.

Definition 3.1. A topological space (V, τ) is termed graphic if there exists a graph G such that it satisfies the necessary structural conditions($\tau = \tau_G$)

Example 3.2. The discrete topology on the set $V = \{v_1, v_2, v_3, v_4, v_5\}$ is the graphic topology that is generated by C_5

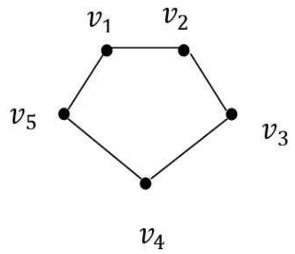

Figure (1) The graph C_5.

For every vertices v_i in C_5, we have:

$A_{v_1} = \{v_2, v_5\}$, $A_{v_2} = \{v_1, v_3\}$, $A_{v_3} = \{v_2, v_4\}$, $A_{v_4} = \{v_3, v_5\}$, $A_{v_5} = \{v_4, v_1\}$

Thus, $\mathcal{S}_{C_5} = \{A_{v_1}, A_{v_2}, A_{v_3}, A_{v_4}, A_{v_5}\}$, and $\mathcal{B}_{C_5} = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}$

So, the topology generated by C_5 is $\tau_{C_5} = \mathcal{P}(V)$, which shows that τ_{C_5} is a discrete topology on V.

Remark 3.3. It is clear for any graph G, the graphic topology is not indiscrete, because $A_{\mathrm{v}} \neq V$ for every $\mathrm{v} \in V$; therefore, the indiscrete topological space is not graphic.
In graph G, for each $x \in V$, U_x is defined as the intersection of all open sets containing a given point or subset defines its smallest surrounding topology a given set or point within the space. x, is defined as the smallest open set that contains a given set or point x, it is easy to check that the family $\{U_x : x \in V\}$ represents a basis for the topology τ_G, moreover, it is contained in any other basis, this is what is mentioned in the following definition;

Definition 3.4. if $G = (V, E)$ be a graph. For each $x \in V$, the family $\mathcal{M}_G = \{U_x : x \in V\}$ is the minimal basis for the topological space (V, τ_G).

Proposition 3.5. Let $G = (V, E)$ be a graph, U_x is defined as the intersection of all open sets that include a given set or point within the space. x then
i. $U_x = \bigcap_{y \in A_x} A_y$
ii. $z \in U_x$ if and only if $A_x \subseteq A_z$
Proof.
i. Since $\mathcal{S}_G$ is the sub-basis of τ_G and U_x is an open set, then $U_x = \bigcap_{y \in W} A_y$ for some $W \subseteq V$. This means that $x \in A_y$ for all $y \in W$. Hence, $y \in A_x$ for all $y \in W$, therefore $W \subseteq A_x$ and so $x \in \bigcap_{y \in A_x} A_y \subseteq U_x$. From the definition of U_x, we have $U_x = \bigcap_{y \in A_x} A_y$ which is complete the proof.
ii. from (i), we have $z \in U_x$ equivalent to $z \in \bigcap_{y \in A_x} A_y$, this means that, for all $y \in A_x$, $z \in A_y$ then $y \in A_z$ and so $A_x \subseteq A_z$. □

the following proposition describes the minimal open sets for the graphic topology;

Proposition 3.6.[13] For any $x, y \in V$ in a graph $G = (V, E)$, we have
i. $U_x \cap A_x = \emptyset$, thereover $U_x \subseteq A_x{}^c$
ii. if x adjacent y, then $U_x \cap U_y = \emptyset$

Example 3.7. The graph $G = (V, E)$ as in Figure 2, whose vertices set is $V = \{a, b, c, d, e\}$

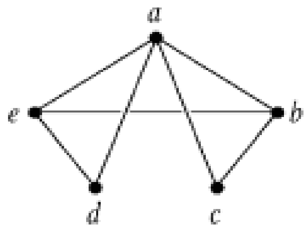

Figure (2) The graph G.

We have,

$$A_a = \{b, c, d, e\}, A_b = \{a, c, e\}, A_c = \{a, b\}, A_d = \{a, e\}, A_e = \{a, b, d\}$$

Thus, $U_a = \{a\}$, $U_b = \{b\}$, $U_c = \{c, e\}$, $U_d = \{b, d\}$, $U_e = \{e\}$

Therefore, $\mathcal{M}_G = \{\{a\}, \{b\}, \{c, e\}, \{b, d\}, \{e\}\}$ is the minimal basis for the topology

$$\tau_G = \left\{ \begin{array}{c} \emptyset, V, \{a\}, \{b\}, \{e\}, \{a, b\}, \{a, e\}, \{b, e\}, \\ \{c, e\}, \{b, d\}, \{a, b, e\}, \{a, c, e\}, \{a, b, d\}, \\ \{b, d, e\}, \{a, b, c, e\}, \{a, b, d, e\}, \{b, c, d, e\} \end{array} \right\}$$

The terms of the previous proposition can be easily verified through this example. .

Remark 3.8. Note that if U_x is maximal in $\mathcal{M}_G$ then A_x is minimal in $\mathcal{S}_G$ and vice versa.

Proposition 3.9 [7]. Let (V, τ) be a finite topological space of order n. if $|U_x| = n$ for some $x \in V$, then (V, τ) is not graphic.

Corollary 3.10 For any topological space (V, τ), if there exists $x \in V$ such that $U_x = V$, then (V, τ) is not graphic, on the other hand, if for every $x \in V$, $U_x = \{x\}$ it means that (V, τ) is a discrete graphic topological space.

Definition 3.11 Two graphs $G_1 = (V_1, E_1)$, and $G_2 = (V_2, E_2)$ are said to be isometric, and written $G_1 \approx G_2$, if there exists a bijection $\varphi: V_1 \longrightarrow V_2$ with $(x, y) \in E_1$ if and only if $(\varphi(x), \varphi(y)) \in E_2$ for any two vertices $x, y \in V_1$, the function φ is called an isomorphism

Remark 3.12 It is clear that, if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic, then the graphic topological spaces (V_1, τ_{G_1}), and (V_2, τ_{G_2}) are homeomorphic. The converse is not true always, as an example, the graphic topologies τ_{C_n} and τ_{K_n} for $n > 4$ are homeomorphic which are discrete, but C_n and K_n are not isomorphic graphs.

4. Connectedness

In connected graph, there exists a path between every pair of its vertices. Here our purpose is to give special cases that affect the connectivity of a graphic topology , and also to give certain condition on the adjacency sets in a graphic space that implies that the graph is connected.

Theorem 4.1. The graphic topological space (V, τ_G) of any disconnected graph $G = (V, E)$ is disconnected.

Proof. Let the graph G be disconnected, and $G_i = (V_i, E_i), (i = 1, \dots n)$ be the connected subgraphs of G, such that $\{G_i : i = 1, \dots n\}$ is set of all components of G, we have $V_i = \bigcup_{x \in V_i} A_x$, therefore $V_i \in \tau_G$ for all $i = 1, \dots n$. And since $(V_i)^c \subset V$ because it is the union of vertices of other components, this means $V = {V_i}^c \cup V_i$, thus V is union of two non-empty disjoint open subsets in τ_G. Hence the space (V, τ_G) is disconnected. □

In general, the converse of the last theorem is not true, since the graphic topology of C_n is discrete, which is disconnected but the graph C_n is connected.

Proposition 4.2 The graphic topological space (V, τ_G) of a graph $G = (V, E)$ is disconnected if $U_x = \{x\}$ for each $x \in V$.

Proof. If $U_x = \{x\}$ for each $x \in V$ in the graph G, then τ_G is discrete topology and so it is disconnected. □

Proposition 4.3 [13]. Let $\mathrm{G} = (V, E)$ be a graph and (V, τ_G) is the graphic topological space of V. If there is a vertex $\mathrm{v} \in V$ with U_{v} is maximal and minimal in $\mathcal{M}_G$, then the graphic topological space (V, τ_{G}) is disconnected.

Proposition 4.4 Let $\mathrm{G} = (V, E)$ be a graph and (V, τ_G) is the graphic topological space of V. If (V, τ_G) is connected, then for each $\mathrm{v} \in V$, there exist two vertices $\mathrm{u}, \mathrm{w} \in V$ such that $A_u \subset A_{\mathrm{v}} \subset A_w$.

Proof. Suppose that (V, τ_G) is connected, then for every vertex $\mathrm{v} \in V$, we have U_{v} is not maximal nor minimal. First, we assume that U_{v} is not maximal, hence the A_{v} is not minimal so there is $A_u \subset A_{\mathrm{v}}$ for some $\mathrm{u} \in V$, secondly, let U_{v} be not minimal, this implies that A_{v} is not maximal which means there is $\mathrm{w} \in V$ and $A_{\mathrm{v}} \subset A_w$, this proves the assertion.

Example 4.5 Consider the cycle graph C_5 in Figure 1. the sets $A_{v_1} = \{v_2.v_5\}$, $A_{v_2} = \{v_1.v_3\}$, $A_{v_3} = \{v_2.v_4\}$, $A_{v_4} = \{v_4.v_5\}$, $A_{v_5} = \{v_1.v_4\}$, are no containment between them, so the graphic topology τ_G is disconnected.

Remark 4.6 The graphic topology (V, τ_G) is irreducible if and only if the graph G is empty, because, if there is any edge $e = (\mathrm{u}, \mathrm{v})$ in the graph G, then from proposition (3.6) we have the open sets U_{u} , U_{v} are disjoint, but for irreducibility, each open sets must be intersect.

Graph G that is disconnected, has connected components, while the connected graph has only one connected component, in this sense the authors in paper (8) defined a new topology on connected components of a graph, which is called $\mathcal{Z}_G$-graphic topology, and it is defined as follows:

Definition 4.7 [8]**.** Let $\mathrm{G} = (V, E)$ be a graph, and $A \subset V$. $A \in \mathcal{Z}_G$ if it is a connected component of G.

From the previous definition, It is clear that $\mathcal{Z}_G \subset \tau_G$, that is connected if and only if it is indiscrete.

Theorem 4.8 [8]**.** Let $\mathrm{G} = (V, E)$ be a graph, the graph G is connected if and only if $\mathcal{Z}_G$ is a connected topology on V.

Remark 4.9 From the above, we can say that the graph G is connected if and only if its $\mathcal{Z}_G$-graphic topology is indiscrete.

By defining the $\mathcal{Z}_G$-graphic, the researcher in [8] was able to give an answer to the question posed in [13], which is: What are the necessary and sufficient conditions for connectivity of the graphic topology?. In this work, we present another answer with a different perspective to the sufficient conditions in question posed.

Proposition 4.10 Let $\mathrm{G} = (V, E)$ be a graph and (V, τ_G) is the graphic topological space of V**.** If for every distinct vertices $\mathrm{v}, \mathrm{u} \in V$ we have $A_{\mathrm{v}} \cap A_{\mathrm{u}} \neq \emptyset$. Then $\mathcal{Z}_G$ $-$graphic space is an indiscrete.

Proof.

By contradiction, let $\mathcal{Z}_G$ $-$graphic space is not indiscrete, then there are $A_1, A_2 \in \mathcal{Z}_G$ which are components in the graph G. so they are open in the space (V, τ_G)**.** From the given condition, we have a vertex $w \in V$ such that $w \in A_1 \cap A_2$, which is a contradiction, because the components are disjoint. □

Corollary 4.11 The non-empty intersection property of adjacency sets in a graphic topological space (V, τ_G) implies that the graph G is connected.

5. Conclusion

In this paper definition and properties of graphic topology have been considered, along with the connection between graphic topology and undirected locally finite graphs, and also shown that there are topological spaces that are not graphic as the indiscrete space, addition the investigation of connectedness and irreducibility of graphic topology. Finally, cases that do not meet with the connectedness condition have been studied, and special conditions have been presented on adjacency sets in graphic topologies that ensure connectivity of the graph.

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