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# Least Square Method for Solving Second-Order Eigenvalue Problems with Nonlinear Delay Involves Ordinary Differential Equations with Nonlinear Delay

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## طريقة المربعات الصغرى لحل مسائل القيم الذاتية من الدرجة الثانية ذات التأخير الغير خطي تتضمن معادلات تفاضلية عادية ذات التأخير

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**Abstract:**
The objective of this research is to survey nonlinear eigenvalue problems of the second order with delay, which are considered one of the known expansion methods, involving ordinary differential equations with delay. To solve this type of problem, one of the known expansion methods known as the least squares method, will be advanced.

**Keywords**: Delay Eigenvalue Problems, Least Squares Method, Nonlinear Second- Order Sturm Liouville Problems.

**الملخص:**
يهدف هذا البحث إلى دراسة مسائل القيم الذاتية غير الخطية من الدرجة الثانية مع التأخير، والتي تُعد من طرق التوسع المعروفة، حيث تتضمن المعادلات التفاضلية العادية مع التأخير. لحل هذا النوع من المسائل، سيتم تطوير إحدى طرق التوسع، والمعروفة بطريقة المربعات الصغرى.

**الكلمات المفتاحية:** مسائل القيم الذاتية ذات التأخير، طريقة المربعات الصغرى، مسائل ستورم ليوفيل الغير خطية من الدرجة الثانية.

**Introduction:**
Delay differential equations are crucial for solving problems in life and have several uses in other fields of study, including biology, physics, engineering, mechanics, and economics. These equations were used in the theory of automated control by Nadia in [6] as a mathematical application.
Nonlinear eigenvalue problems of the second-order are characterized by nonlinear ordinary differential equations accompanied by specific boundary conditions defined over a given time interval. These problems arise frequently in a wide range of scientific and engineering applications. A notable variant of these problems involves time delays, giving rise to what is known as the delayed nonlinear eigenvalue problem.[5]

This class of problems belongs to a broader category of spectral problems, distinguished by the presence of delay terms, which lead to unique behaviors in the associated eigenvalues and eigenfunctions.[1]
In this study, we address such delayed eigenvalue problems and propose a solution approach based on the least squares method.

**Fundamental Definitions and Points**
Some fundamental definitions and statements that are necessary for this work are recalled in this section.We begin with the definition that follows.

**Definition:**
A delay differential equation is an equation used to evaluate an unknown function and some of its derivatives at different points, using a fixed number or function of values. The n-th order linear delay differential equation with a single constant delay can be defined as follows:

$$\varphi\big(u, f(u), f(u-t_1), \ldots \ldots . f(u-t_m), f'(u), f'(u-t_1), \ldots, f'(u-t_m), f^n(u), \ldots, f^n(u-t_m)\big) = h(u), \quad u \in [a,b] \qquad [1.1]$$

Where , h are a given function and $t_1$,$t_2$ ,………..,$t_m$are given fixed positive numbers called the time delely. [5]

In this study, we refer to equation [1.1] as a homogeneous delay differential equation when $h(u) \geq 0$; otherwise, it is referred to as a non-homogenous delay differential equation. [5]

**Definition:**
When the delay differential equation is non-linear concerning the unknown function that includes different inputs and their associated derivatives present within it, it is categorized as non-linear.[2]
Therefore, the following definition provides the novel notions of this work.

**Definition:**
When the delay eigenvalue problem, which consists of a delay ordinary differential equation, is nonlinear with regard to the unknown eigenfunction entered with various propositions and their differential consequences that emerged in it, it is considered nonlinear.

Next, look at the nonlinear delay second-order eigenvalue problem that follows:

$$[1.2] -\big(q(u)f'(u)\big)' + z(x)f(u-t) - h\big(u, \lambda, f(u-t)\big) = 0$$

with the accompanied conditions:

$$\omega_1 f(\omega) + \omega_2 f'(\omega) = 0 \,, u \in [\omega - t, \omega]$$

$$[1.3] \beta_1 f(\beta) + \beta_2 f'(\beta) = 0 \,, u \in [\beta - t, \beta]$$

$$f(u-t) = E(u-t), \text{if } u - t < \omega$$

Let , $\beta_1$ , $\beta_2$ ,$\omega_1$ , $\omega_2$ , q , q 'and z be real-valued continuous functions defined on the closed interval [ω, β], where q(u)>0 for all u∈ [ω, β]. It is assumed that not both coefficients in the boundary conditions are zero, ensuring the problem is well-posed.
We consider a delay parameter t>0, and a nonlinear function h that is well-defined with respect to the unknown function f. The initial function E is given on the interval $u \in [u_0 - t\,,\ u_0]$, serving as the history function due to the time delay.
The objective is to determine the eigenvalue λ for which a nontrivial solution f exists satisfying the differential equations given in [1.2] and [1.3] . In this setting, λ is termed the delay eigenvalue, and the corresponding solution f is referred to as the delay eigenfunction.

In other words, the function f is regarded as an eigenfunction with respect to the spatial variable u, and the nonlinear function h(u, λ, f(u−t)) introduces a dependence on both the delay and the eigenvalue parameter.
The structure of the problem defined by equations [1.2] and [1.3] shares fundamental characteristics with classical second-order linear eigenvalue problems, but with the additional complexity introduced by the presence of both nonlinearity and time delay. [2]

**Observations:**

1-The linear delay operator: $L=-\frac{d^2}{du^2}q(u) - \frac{d}{du}q(u) + R(u)z(u)$ , where $R(u)$ is an operator defaine by $R(u)f(u) = f(u-t)$ , is self – adjoint [3].

2-The eigenfunctions of the delay are perpendicular.[8]

3- There are an infinite number of eigenvalues with delay that form an increasing sequence, where $\lambda_i$ approaches infinity as i approaches infinity. In addition, the eigenfunctions associated with these exact eigenvalues i have roots within the interval (ω, β ) .

4- In $L^2[\omega, \beta]$, the delay eigenfunctions constitute full and normal system, making them suitable for use as a basis in functional expansions.

Only one delay eigenfunction in $L^2[\ \omega\ ,\ \beta\ ]$ corresponds to each delay eigenvalue. 5- Refer to [1] to verify observations (2-4).

**The Least-Square Method**

One widening technique for solving linear and nonlinear differential equations and equations with or without delays is The Least-Square Method.[3],[9]
Here, we refine this approach to address the issue raised by equations [1.2] and [1.3].

The function f, which is initially unknown, is expressed in terms of a linear combination of n linearly independent functions to facilitate its approximation $\{Q\ j\}^n$ j=1 is the foundation of the technique, that is write:

$$f(u) = \sum_{j-1}^{n} Q_j(u) \qquad [2.1]$$

Which means that, $f(u-t) = \sum_{j=1}^{n} Q_j(u-t)$

To obtain a new approximated solution, this approximated solution needs to meet the boundary conditions provided by equations [1.3]. This approximated solution can be intervened into equation [1.2] to get:

$$[2.2]A(u,\lambda,\vec{s}) = -(p(u)\sum_{j=1}^{n} Q_j'(u))' + z(u)\sum_{j=1}^{n} Q_j(u-t) - h(u,\lambda,\sum_{j=1}^{n} Q_j(u-t))$$

Where A is the error in approximation of equation [2.2] and $\vec{s}$ is the vector of n-2 elements of $s_j$ , j = 1,2,3,………..n , [7] , [9] .

Therefore, in order to reduce the functional

$$[2.3]\mu(\lambda,\vec{s}) = \int_a^b (A(u,\lambda,\vec{s}))^2 du$$

Put $\frac{\partial\mu}{\partial\lambda} = \frac{\partial\mu}{\partial s_j} = 0,\ \ j = 1,2,3,\dots\dots,n$

To obtain a system of n–1 nonlinear equations with n–1 unknowns that can be resolved using any appropriate technique to determine the values of s and λ [3] , [7].
To verify this method, look at the following examples:

**Example**

Solve the nonlinear delay eigen-value problem that follows:

. $$[2.4] -\left(u\, y'(u)\right)' + 2uy(u-1) - \lambda(y^2(u-1) - 0.5) = 0\,, u \in [1,2]$$

with the corresponding boundary conditions:

, $u \in [0,1] y(1) = y'(1)$

, $u \in [1,2] y(2) = 2\, y'(2)$

$$[2.5] y(u-1) = u - 1$$

To attempt to solve the problem, we employ the least-squares method. This is accomplished by approximating the unknown function y as a degree three polynomial, and writing:

$$y(u) = \sum_{j=1}^{4} s_j u^{j-1}$$

$$y(u) = s_1 + s_2 u + s_3 u^2 + s_4 u^4$$

Therefore,

$$y(u-1) = s_1 + s_2(u-1) + s_3(u-1)^2 + s_4(u-1)^4$$

However, the boundary conditions provided by equations [2.5] must be met by this approximated solution, therefore it reduces to:

$$s_1 = s_2$$

$$s_3 = -6s_4$$

Then, $y(u) = s_1 + s_1 u - 6s_4 u^2 + s_4 u^3$

From which, we have:

$$y(u-1) = s_1 + s_1(u-1) - 6s_4(u-1)^2 + s_4(u-1)^3$$

By substituting this approximated solution into equation [2.4], we obtain

$$\begin{aligned} A(u,\lambda,s_1,s_4) = {} & -u(-18s_4 + 6s_4 u) - (s_1 - 18s_4 u - 9s_4 + 3s_4 u^2) \\ & + 2u(s_1 + s_1(u-1) - 6s_4(u-1)^2 + s_4(u-1)^3) \\ & - \lambda[(s_1 + s_1(u-1) - 6s_4(u-1) + s_4(u-1)^3)^2 - 0.5] \end{aligned}$$

Thus,if we minimize the functional:

$$\mu(\lambda, s_1, s_4) = \int_1^2 (A(u,\lambda,s_1,s_4))^2 du$$

Set $\frac{\partial\mu}{\partial\lambda} = \frac{\partial\mu}{\partial s_1} = \frac{\partial\mu}{\partial s_4} = 0$ to get the following system of nonlinear equations:

$$\frac{\partial}{\partial\lambda}\int_1^2 (A(u,\lambda,s_1,s_4))^2\, du = 0$$

$$\frac{\partial}{\partial s_1}\int_1^2 (A(u,\lambda,s_1,s_4))^2\,du = 0$$

$$\frac{\partial}{\partial s_4}\int_1^2 (A(u,\lambda,s_1,s_4))^2\,du = 0$$

Using any appropriate approach to solve the given system, we may determine that the nontrivial solution is $\lambda$ = 2, $s_1$= 1, and $s_4$= 0. Consequently, the associated delay eigenfunction and delay eigenvalue are 2.

$y(u-1) = 1 + (u-1)$ , $u\epsilon[1,2]$

On the whole, the same outcome can be achieved for any values of n, n$\epsilon$N, if

$$y(u) = \sum_{j=1}^{n} s_j u^{j-1}$$

That is if $y(u-1) = \sum_{j=1}^{n} s_j (u-1)^{j-1}$ , then $y(u-1) = 1 + (u-1)$ , $u\epsilon[1,2]$

matching the exact same delay eigenvalue.

**Example**

Solve the nonlinear delay eigenvalue problem that follows

$y''(u) + \lambda\, y(u) = \sin(\pi u)$ , $u \in [0,1]$ [2.6]

with the corresponding boundary conditions:

$y(0) = 0$

$y(1) = 0$

We assume that $y(u)$ can be expressed linearly in terms of a set of basis functions as follows:

$$y(u) = \sum_{n=1}^{N} s_n \sin(n\pi u)$$

We differentiate $y(x)$

$$y'(u) = \sum_{n=1}^{N} s_n n\pi \cos(n\pi u)$$

$$y''(u) = \sum_{n=1}^{N} - s_n n^2\pi^2 \sin(n\pi u)$$

Substituting in to differential equation [2.6] we obtain :

$$-\sum_{n=1}^{N} s_n n^2\pi^2 \sin(n\pi u) + \lambda \sum_{n=1}^{N} s_n n\pi \cos(n\pi u) = \sin(\pi u)$$

Rewrite the equation to obtain:

$$\sum_{n=1}^{N} s_n(-n^2\pi^2 + \lambda)\sin(n\pi u) = \sin(\pi u)$$

To find $\lambda$ and $s_n$, we use the least squares method.

Start by calculating the error:

$$\mu = \int_0^1 \left(\sum_{n=1}^{N} s_n(-n^2\pi^2 + \lambda)\sin(n\pi u) - \sin(\pi u)\right)^2 du$$

Set $\frac{\partial\mu}{\partial\lambda} = \frac{\partial\mu}{\partial s_n} = 0$

After performing the calculus, we end up with a system of equations. for $n = 1$ , for example ,we find:

$$\int_0^1 (-\pi^2 + \lambda)\sin^2(\pi u)du = 0$$

Where:

$$\int_0^1 \sin^2(\pi u)du = \frac{1}{2}$$

Thus

$$(-\pi^2 + \lambda)\frac{1}{2} = 1$$

Then $\lambda = \pi^2 + 2$

We then substitute the value of $\lambda$ into the original equation. We ensure that all boundary conditions are met.

Assume we want to find $s_1$

$$s_1 = \frac{\int_0^1 \sin(\pi u)\sin(\pi u)\, du}{\int_0^1 \sin^2(\pi u)du} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

**Conclusions:**

From this research, we can infer that :

1. The second-order eigenvalue problems with nonlinear delays, which include nonlinear ordinary differential equations with delays, exhibit the same characteristics as those involving nonlinear ordinary differential equations, regardless of whether they include delays or not.
2. These kinds of problems can be resolved by developing the least squares method.

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