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Van Kampen Diagrams and Pictures: Definitions, Operations, and Conversions

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مخططات وصور فان كامبين: تعريفاتها، عملياتها، والتحويل بينها

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Abstract
Van Kampen diagrams are an important and powerful tool in many areas of mathematics and engineering. These diagrams are used to visually and intuitively represent the relationships and interactions between different elements, making complex structures easier to understand and analyze. Van Kampen diagrams are distinguished by their ability to analyze structures consisting of multiple elements and identify common points between them. In this paper, we initially explained Van Kampen diagrams and showed two ways of finding them. Subsequently, we explained Van Kampen pictures and demonstrated four ways of obtaining them. Finally, using various methods, we were able to show how these diagrams can be converted into pictures and vice versa.

Keywords: Van Kampen diagram, Van Kampen Pictures, Operations.

الملخص
تُعتبر مخططات فان كامبين أداة هامة وقوية في العديد من مجالات الرياضيات والهندسة. تُستخدم هذه المخططات لتمثيل العلاقات والتفاعلات بين العناصر المختلفة بشكل بصري وبديهي، مما يسهل فهم الهياكل المعقدة وتحليلها. وتتميز مخططات فان كامبين بقدرتها على تحليل الهياكل التي تتكون من عناصر متعددة وتحديد النقاط المشتركة بينها.
في هذه الورقة، وضحنا في البداية مخططات فان كامبين وبيّنا طريقتين لكيفية إيجادها. ثم قمنا بشرح صور فان كامبين وعرضنا أربع طرق للحصول عليها. وأخيرًا، تمكنا، باستخدام طرق مختلفة، من إظهار كيفية تحويل هذه المخططات إلى صور والعكس.

الكلمات المفتاحية: مخططات فان كامبين، صور فان كامبين، العمليات.

1. Introduction

In the mathematical area of geometric group theory, a Van Kampen diagram (sometimes also called a Lyndon-Van Kampen diagram) is a planar diagram used to represent the fact that a particular word in the generators of a group given by a group presentation represents the identity element in that group.

2. History

The notion of a Van Kampen diagram was introduced by Egbert Van Kampen in 1933.

This paper appeared in the same issue of American Journal of Mathematics as another paper of Van Kampen, where he proved what in now known as the seifert-van Kampen theorem. The main result of the paper on Van Kampen diagram, now known as the Van Kampen lemma can be deduced from the seifert-Van Kampen theorem by applying the latter to the presentation complex of a group.

Currently Van Kampen diagrams are a standard tool in geometric group theory. They are used, in particular, for the study of isoperimetric functions in groups, and their various generalizations such as isodiametric functions, filling length function, and so on.

3. Van Kampen diagram

Definition 1

A group presentation is a pair $P = \langle X | r \rangle$ where

i. X is a generating set.
ii. r is a set of cyclically reduced words on X.

Set r is a called the set of relators.

A presentation P is finite if X and r are both finite.

Example 1:

i. $P_1 = \langle x | x^6 \rangle$
ii. $P_2 = \langle a, b | a^2, b^3, aba^{-1}b^{-1} \rangle$
iii. $P_3 = \langle a, b | a^2, b^3 \rangle$
iv. $P_4 = \langle a, b | a^2, a\ b\ a^{-1}b^{-1} \rangle$

Sometimes relators are written in term of relations.

For example, in ii. we may write

$$P_2 = \langle a, b | a^2 = 1, b^3 = 1, aba^{-1}b^{-1} = 1 \rangle$$

Definition 2

Let $P = \langle X | r \rangle$ be a group presentation.

A (Van Kampen) diagram D over P is a graph such that

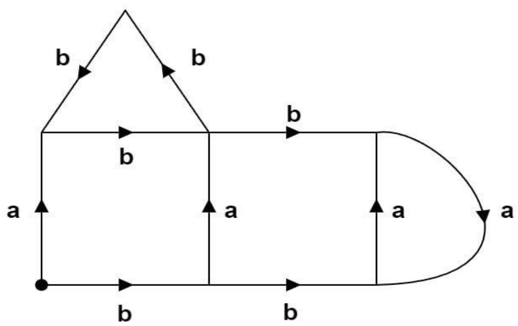
i. Edges are labeled by elements of X such that.
ii. Travelling along any face from a suitable vertex of the face will read an element of r (clockwise or anticlockwise).
iii. A base point a boundary vertex.

Label of D (denoted by W(D)) is the word at the boundary label when we read clockwise.

Example 2:

$$P = \langle a, b | a^2, b^3, aba^{-1}b^{-1} \rangle$$

i. Edges are labeled by $a\ or\ b$.
ii. Face read $a^2, b^3\ or\ aba^{-1}b^{-1}$



W(D)= $ab^{-2}bab^{-2}$

Definition 3

There are two operations on diagrams.

Sew:If there exist two adjacent boundary edges with same label but in opposite direction, then sew both edges.

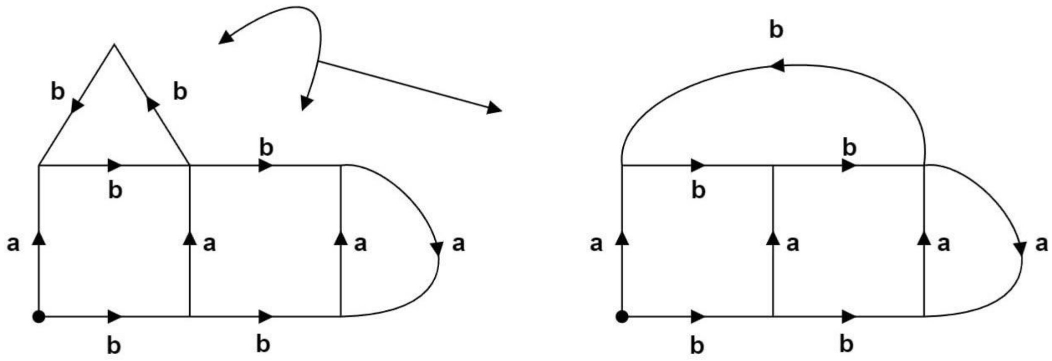
Unsown: opposite of sew.

Draw in a way you feel comfortable!

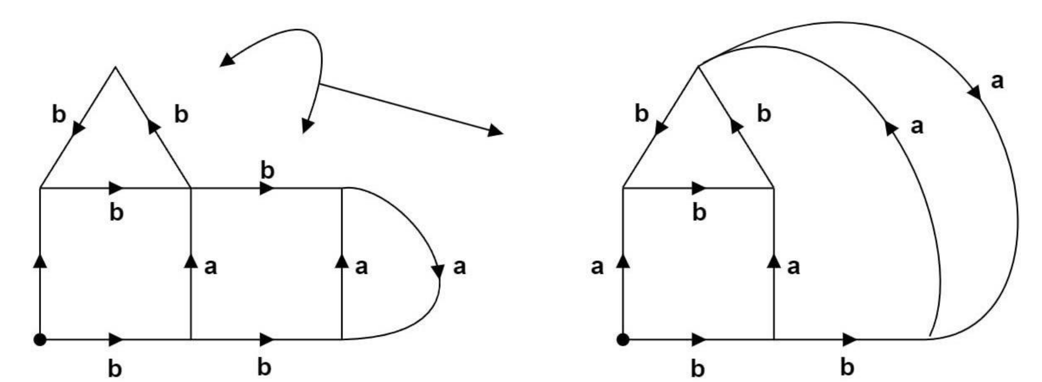
Shapes of these diagrams may be different but they have same faces and same boundary label.

Example 3:

$$P = \langle a, b | a^2, b^3, aba^{-1}b^{-1} \rangle$$



or



Definition 4

Let $P = \langle X|r\rangle$ be a group presentation.

Two diagrams B and D over P are equivalent if D is obtainable from B using a finite number of sewing operations.

We denote $B \approx D$.

As usual the set of all equivalent diagrams containing D will be denoted by [D].

It is quite difficult to form groups from diagrams and so we will use the dual of diagrams namely pictures!

Theorem 1: (Van Kampen Lemma)

Let $P = \langle X|r\rangle$ be a presentation defining the group $G(P)$. If W is a word on X, the

$$W \approx 1 \Leftrightarrow W \approx^1 \left(U_1 R_1^{\varepsilon_1} U_1^{-1}\right)$$

$$\left(U_2 R_2^{\varepsilon_2} U_2^{-1}\right) \dots \left(U_n R_n^{\varepsilon_n} U_n^{-1}\right)$$

Where $R_i^{\varepsilon_i} \in r$, $\varepsilon = \pm 1$, U_i word on X.

Proof

Clearly

$$W \approx 1 \Leftrightarrow W \approx^1 \left(U_1 R_1^{\varepsilon_1} U_1^{-1}\right)$$

$$\left(U_2 R_2^{\varepsilon_2} U_2^{-1}\right) \dots \left(U_n R_n^{\varepsilon_n} U_n^{-1}\right) \Longrightarrow W \approx 1$$

conversely, use induction on n.

Example 4:

consider the presentation $\langle a, b|a^2, b^3, aba^{-1}b^{-1}\rangle$

i. Show that $abab^{-1} \approx 1$.
ii. Write the van Kampen lemma word for $abab^{-1}$.

Solution

i. $abab^{-1} \approx abaa^{-2}b^{-1} \approx aba^{-1}b^{-1} \approx 1$
ii. $abab^{-1} \approx^1 aba^{-1}aab^{-1} = aba^{-1}aab^{-1}$
$\approx^1 aba^{-1}b^{-1}ba^2b^{-1} = (aba^{-1}b^{-1})(ba^2b^{-1})$

Theorem 2: (Van Kampen Lemma)

Let $P = \langle X|r\rangle$ be a presentation defining the group $G(P)$ and W is a word on X. Then $W \approx 1$ if and only if there exists a diagram D over P with boundary label W.

Definition 5

Given a diagram D with boundary label W, it is not really difficult to show that $W \approx 1$.

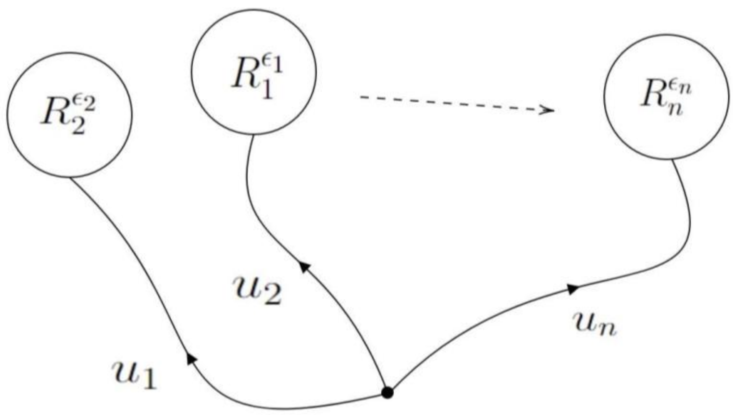
If given $W \approx 1$ then how to draw a diagram with label W?

Using Theorem (1) (Van Kampen Lemma):

$$W \approx 1 \Leftrightarrow W \approx^1 \left(U_1 R_1^{\varepsilon_1} U_1^{-1}\right)\left(U_2 R_2^{\varepsilon_2} U_2^{-1}\right) \dots \left(U_n R_n^{\varepsilon_n} U_n^{-1}\right)$$

Draw *n* balloons with strings!

$$W \approx^1 \left(U_1 R_1^{\varepsilon_1} U_1^{-1}\right)\left(U_2 R_2^{\varepsilon_2} U_2^{-1}\right) \dots \left(U_n R_n^{\varepsilon_n} U_n^{-1}\right)$$



Example 5:

Consider the presentation

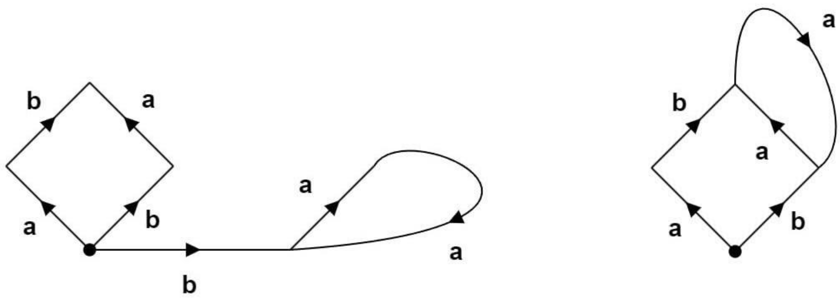
$$\langle a, b | a^2, b^3, aba^{-1}b^{-1} \rangle$$

Draw a diagram with boundary label $abab^{-1}$.

Solution.

$$aba^{-1}b^{-1} \approx (aba^{-1}b^{-1})(ba^2b^{-1})$$

Example (4)



4. Van Kampen Pictures

Definition 6

Let $P = \langle X | r \rangle$ be a group presentation.

A picture $\boldsymbol{P}$ over P is an object consists of

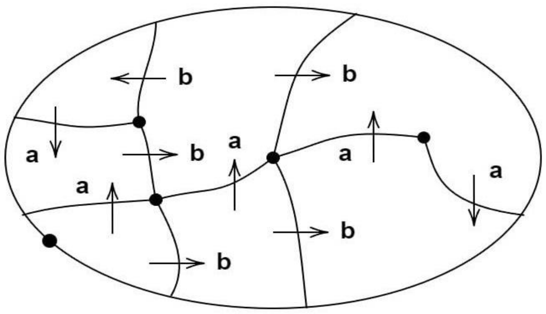
i. A boundary disc with a basepoint.
ii. Discs with label read an element of r (clockwise or anticlockwise).
iii. Disjoints arcs with label element of X.

Label of $\boldsymbol{P}$ (denoted by $W(P)$) is the word at the boundary label when we read clockwise.

Example 6:

$$P = \langle a, b | a^2, b^3, aba^{-1}b^{-1} \rangle$$

i. Arcs are labeled by a or b.
ii. Disce read a^2, b^3 or $aba^{-1}b^{-1}$.



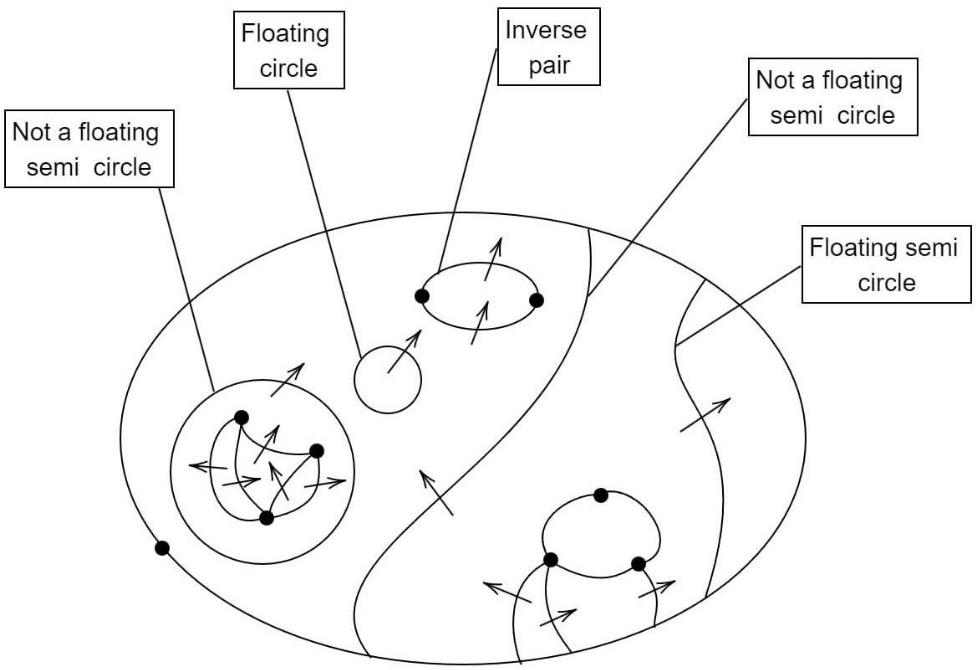
$$W(P) = ab^{-2}bab^{-2}$$

Definition 7

A floating circle is a closed are with on disc inside it.

A floating semi-circle is an arc joining boundary disc such that no disc appears in a face bounded by this circle and the boundary disc.

The inverse pain is a pair of disc joining each other by arcs such none of these arcs joint other discs or joint the boundary disc.



Definition 8

There are four operations on pictures.

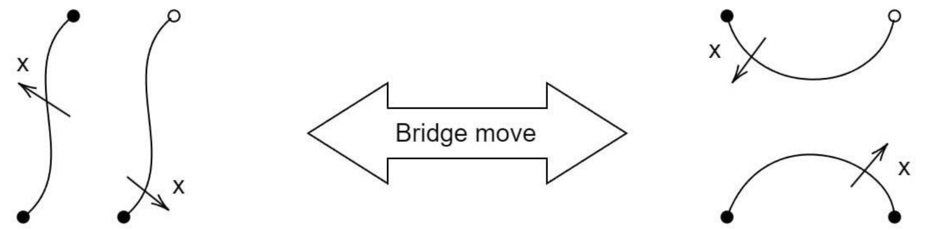
FC: Delete or insert floating circle.

FSC: Delete or insert floating semi-circle.

IF: Delete or insert an inverse pair.

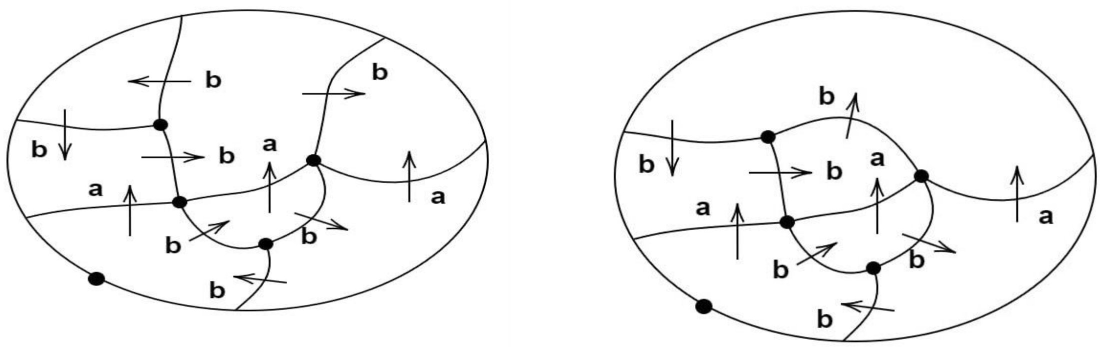
BM: Bridge move.

Twe facing arcs-same label but opposite direction

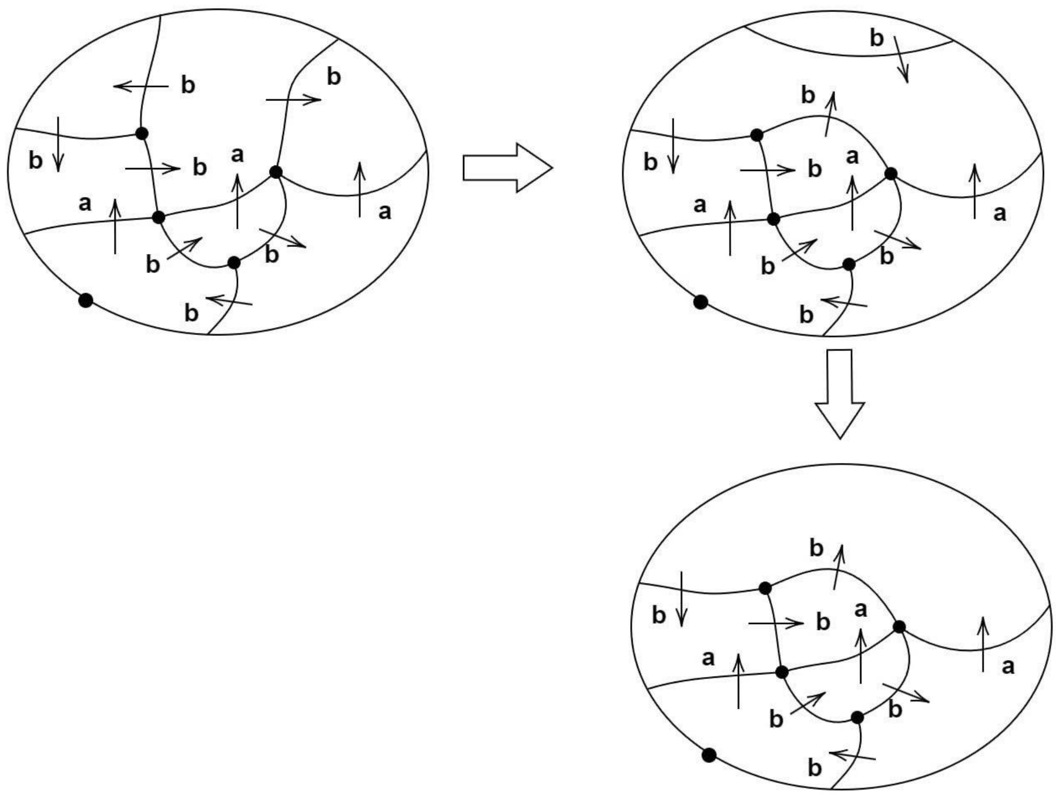


Example 7:

Show in details operations between these pictures.



Solution:



Definition 9

Let $P = \langle X | r \rangle$ be a group presentation.

Two pictures D and Q over P are equivalent if Q is obtainable from P using a finite number of picture operations.

We denote $P \approx Q$.

As usual the set of all equivalent pictures containing P will be $[Q]$.

Theorem 3: (Van Kampen's theorem)

Let $P = \langle X | r \rangle$ be a presentation defining the group $G(P)$ and W is a word on X. Then $W \approx 1$ if and only if there exists a picure P over P with boundary label W.

Definition 10

Given a picture P with boundary label W, it is not really difficult to show that $W \approx 1$.

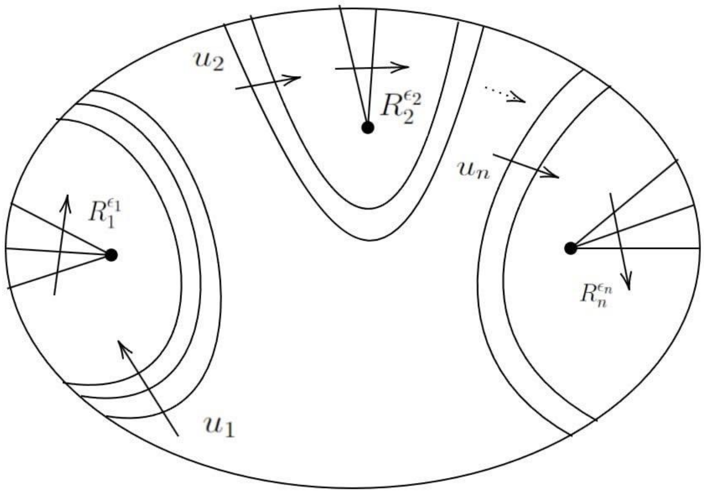
If given $W \approx 1$ then how to draw a picture with label W?

Using Theorem (1) Van Kampen Lemma

$$W \approx 1 \Leftrightarrow W \approx^1 \left(U_1 R_1^{\varepsilon_1} U_1^{-1}\right)\left(U_2 R_2^{\varepsilon_2} U_2^{-1}\right) \dots \left(U_n R_n^{\varepsilon_n} U_n^{-1}\right)$$

Draw *n* curtains!

$$W \approx 1 \Leftrightarrow W \approx^1 \left(U_1 R_1^{\varepsilon_1} U_1^{-1}\right)\left(U_2 R_2^{\varepsilon_2} U_2^{-1}\right) \dots \left(U_n R_n^{\varepsilon_n} U_n^{-1}\right)$$



Example 8:

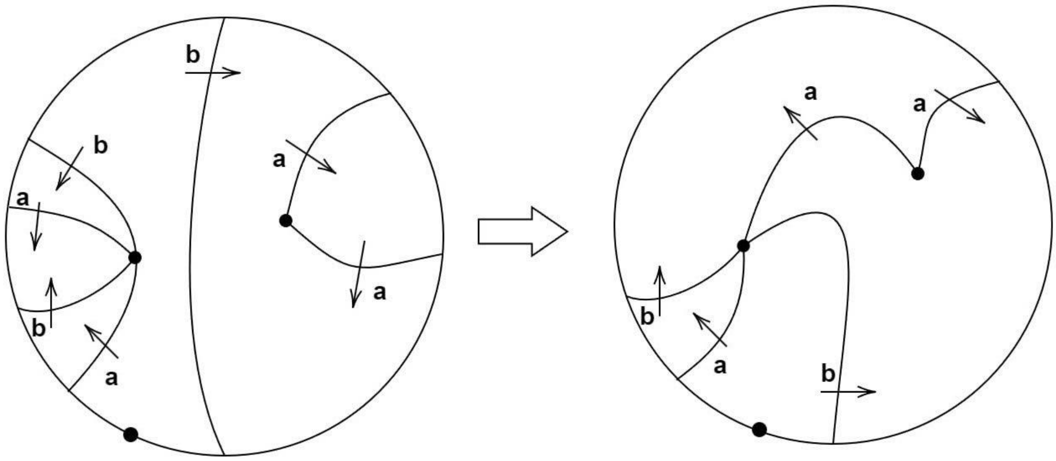
Consider the presentation

$$P = \langle a, b | a^2, b^3, aba^{-1}b^{-1} \rangle$$

Draw a picture with boundary label $abab^{-1}$.

Solution

$abab^{-1} \approx^1 (aba^{-1}b^{-1})(ba^2b^{-1})$ (Example (4)).



Example 9:

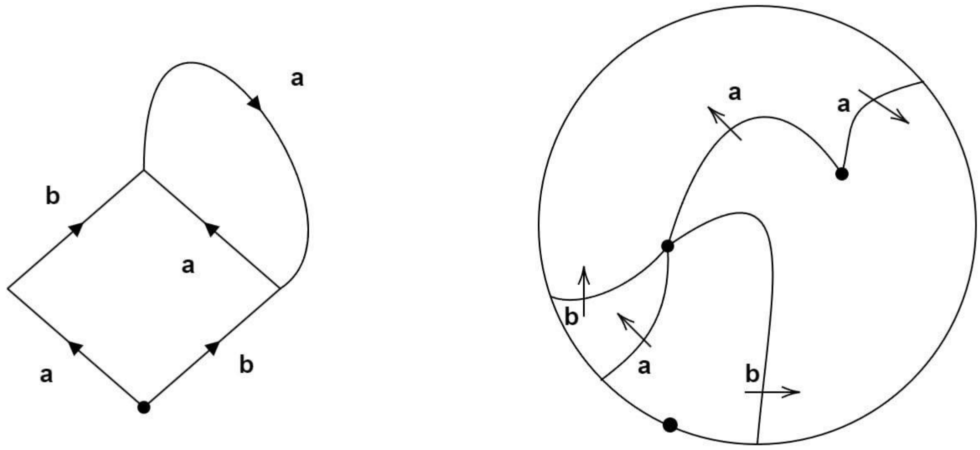
Consider the presentation

$$P = \langle a, b | a^2, b^3, aba^{-1}b^{-1} \rangle$$

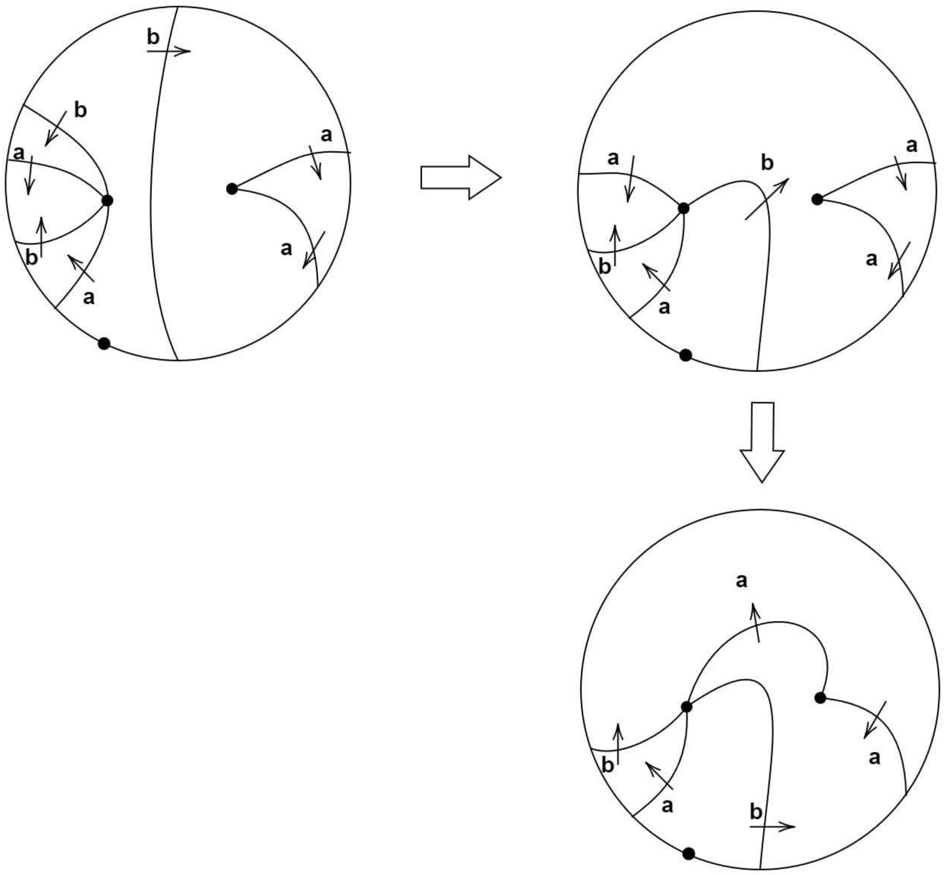
Draw a diagram and a picture with boundary label $abab^{-1}$.

Solution:

From Example (5) and Example (8)



Then



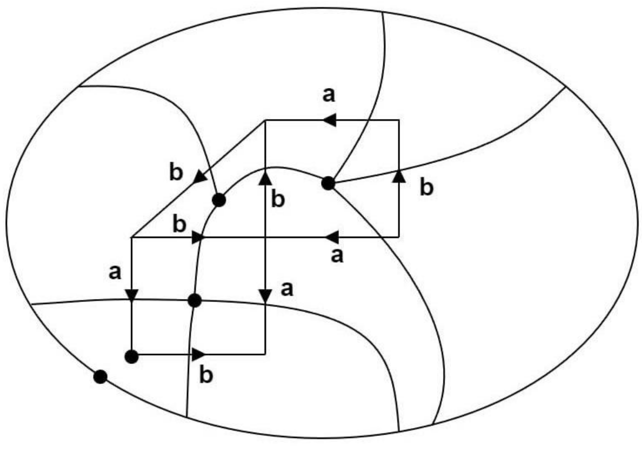
5. Results:

After explaining the diagrams and pictures, we arrived at the following results.

We were able to convert a diagram to a picture and vice versa as follows:

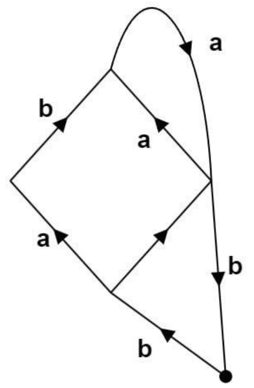
1) From a diagram to a picture:

i. Draw the boundary disc.
ii. Draw a disc for each face.
iii. Joint two adjacent discs every edge they share.
iv. Joint the disc of boundary face to the boundary disc.

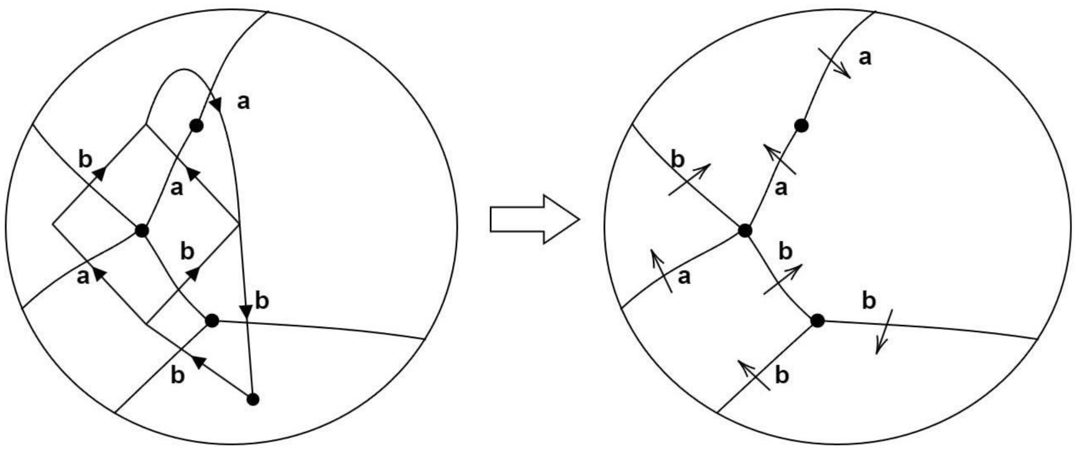


Problem 1:

Draw the picture of this diagram.



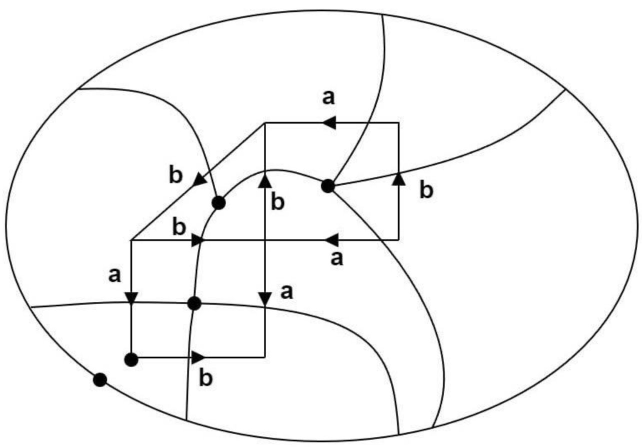
Solution:



2) Form a picture to a diagram:

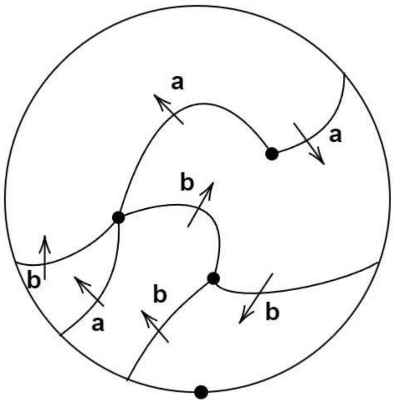
i. Draw a point in each face.

ii. Draw an edge joining two adjacent points.

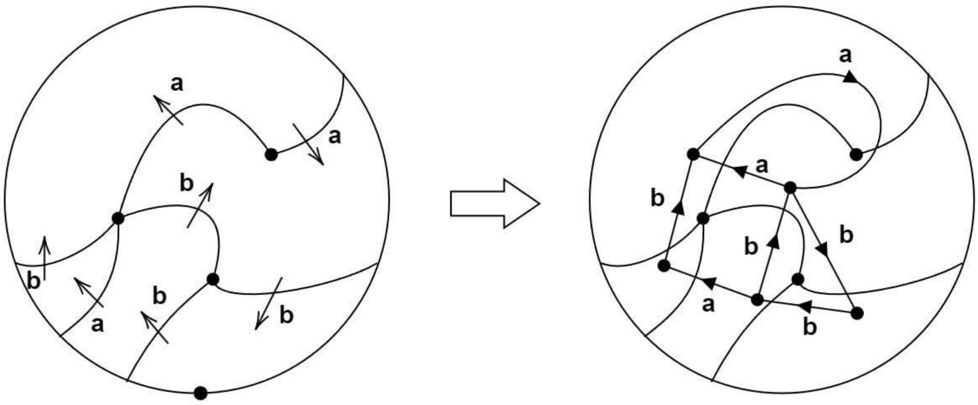


Problem 2:

Draw the dual diagram of this picture



Solution:



6. Conclusions:

In this paper, we presented two operations on diagrams and devised various methods for obtaining Van Kampen diagrams.

Additionally, we introduced four operations on pictures, detailing the

Picture into another. We developed simplified and professional approaches for converting diagrams vice versa, utilizing a range of tools and techniques.

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