African Journal of Advanced Pure and Applied Sciences (AJAPAS)
Online ISSN: 2957-644X
Volume 4, Issue 3, 2025
Page No: 305-319
Website: https://aaasjournals.com/index.php/ajapas/index
ISI 2025: 1.126 **SJIFactor 2024: 6.752** **معامل التأثير العربي: 1.62**

A Comprehensive Examination of Shadow Derivations in Spherically Symmetric Black Holes: From Synge's Historical Insights to Modern Methodologies

Mohamed Emtir Al- Fergani Ali *
Physics Department, Technical Collage of Applied Sciences, AL-Awata, Libya

دراسة شاملة لاشتقاقات الظلال في الثقوب السوداء المتماثلة كرويًا: من رؤى سينج التاريخية إلى المنهجيات الحديثة

محمد امطير الفرجاني علي *
قسم الفيزياء، كلية التقنية للعلوم التطبيقية، العواته، ليبيا

*Corresponding author: mohamedali2013.uk@gmail.com

Received: June 18, 2025	Accepted: August 11, 2025	Published: August 21, 2025

Abstract:
The present work constitutes an erudite discussion of a precise derivation of shadows for spherically symmetric black holes. A concise overview of the historical Synge solution is also furnished, accompanied by a thorough exposition of the contemporary methodology that can be employed with more intricate metrics. The methodology will subsequently be applied to a selection of the most significant black hole solutions. The present work provides a thoroughgoing analysis of black hole shadows for spherically symmetric solutions, with a particular focus on the Schwarzschild and Reissner-Nordström (anti-)de Sitter metrics. The study commences with a historical overview of Synge's pioneering work on the escape of photons from gravitationally intense stars, which laid the foundation for understanding black hole shadows. This employs the technique of Lagrange formalism in order to derive the constants of motion and the trajectory equation for photons within spherically symmetric geometries. This approach facilitates the efficient computation of shadows by determining the turning points of photon orbits and the angular radius of the shadow. The methodology is applied to the Schwarzschild, Reissner-Nordström, and Reissner-Nordström-Kottler solutions, revealing the dependence of the shadow on the black hole's charge and the cosmological constant. The subsequent discourse herein entails an exhaustive examination of its relationship to the frequently employed concepts of escape cone and critical impact parameter. Moreover, the impact of gravitational collapse as well as plasma on the shadow of a black hole is a concomitant consideration in this theoretical framework.

Keywords: black hole shadows, Schwarzschild spacetime, cosmological constant, gravitational collapse, plasma.

الملخص
يُقدم هذا العمل نقاشًا مُعمّقًا حول اشتقاق دقيق لظلال الثقوب السوداء ذات التناظر الكروي. كما يُقدّم لمحةً موجزةً عن حل العالم الفيزيائي سينج ، مصحوبًا بشرحٍ شاملٍ للنهج المعاصر التي يُمكن استخدامها مع مقاييس أكثر تعقيدًا. سيتم تطبيق هذا النهج على مجموعةٍ مُختارةٍ من أهم حلول الثقوب السوداء لاحقا. يُقدّم هذا العمل ايضا تحليلًا شاملًا لظلال الثقوب السوداء للحلول ذات التناظر الكروي، مع تركيزٍ خاص على مقاييس شوارزشيلد ورايسنر-نوردستروم (المضاد) دي سيتر. تبدأ الدراسة بنظرةٍ تاريخيةٍ على عمل سينج الرائد في مجال هروب الفوتونات من النجوم شديدة الجاذبية، والذي وضع الأساس العميق لفهم ظلال الثقوب السوداء. يستخدم هذا التحليل تقنية صيغة لاغرانج لاشتقاق ثوابت الحركة ومعادلة المسار للفوتونات ضمن الهندسة ذات التناظر الكروي. يُسهّل هذا النهج الحسابَ الفعّال للظلال من خلال تحديد نقاط انعطاف مدارات الفوتونات ونصف القطر الزاوي للظل. يطبّق هذه الاسلوب على حلول شوارزشيلد، ورايسنر-نوردستروم، ورايسنر-نوردستروم-كوتلر، كاشفا عن اعتماد الظل على شحنة الثقب الأسود والثابت الكوني. يتضمّن النقاش التالي هنا فحصًا شاملًا لعلاقته

بالمفاهيم المُستخدمة والاكثر شيوعا، وهي مخروط الهروب ومعامل التأثير الحرج. علاوةً على ذلك، يُعدّ تأثير الانهيار التجاذبي، وكذلك تاثير البلازما على ظل الثقب الأسود أحد الاعتبارات المُصاحبة في هذا الإطار النظري.

الكلمات المفتاحية: ظلال الثقب الأسود ، زمكان شوارزشيلد ، الثابت الكوني، الانهيار التجاذبي، البلازما.

Introduction

Black holes represent a particularly fundamental object of study within the realm of general relativity, as evidenced by their classification as a limited set of parameters. Nevertheless, these theories are intriguing and warrant consideration on their own merits, as they illuminate the classical and quantum nature of space-time. Despite the plethora of seminal works that have been published on the theoretical understanding of black holes, observational evidence that is capable of proving their existence has been scarce. The following text has been prepared to provide a thorough survey of the subject matter. The recent observation of gravitational waves from the merging of two black holes [1,2] signifies a substantial advancement in our comprehension of the universe. This finding provides the first evidence for the existence of black holes, a crucial step in our exploration of the cosmos. In addition to the aforementioned points, there is mounting evidence from a variety of astrophysical data [3,4] for the existence of supermassive central objects at the core of numerous galaxies, including our own. This further lends support to the hypothesis of the presence of black holes. Nevertheless, it is important to note that all of the aforementioned evidence for the existence of black holes can be regarded as indirect evidence. This phenomenon can be attributed to the unavailability of a direct observational method for the object in question in celestial bodies. Consequently, a direct observation of the existence of black holes has been evidenced by the observation of their shadow, as was proposed in [5,6] for the ultimate probe of its photon sphere. This has recently been performed by the Event Horizon Telescope (EHT) collaboration [7-12].

It is evident that the image of a black hole, known as the shadow, is merely an representation of the photon sphere. This phenomenon can be attributed to gravitational lensing, an occurrence engendered by the presence of an intensely gravitational field in the immediate vicinity of the black hole. This intense gravitational field causes the photon sphere to be distorted and projected onto the sky as seen by an observer. In summary, the optical appearance of a compact object can be cast into shadow due to the gravitational force exerted by the object itself. The gravitational force exerts a significant influence on all matter, including photons. The phenomenon of light bending is the underlying cause of this distortion, which is the result of the image being formed by the refraction of light. It has been established that photons from any source behind the compact object can cast a shadow on a plane that is observable by an asymptotic observer at infinity. Recent research has revealed that the shadows cast by spherically symmetric black holes tend to be circular in shape. In contrast, the shadows cast by rotating black holes exhibit variability in their circularity, a property that is contingent on the spin parameter, the configuration of the light emission region is located in close proximity to the black hole, and the black hole's angle of inclination is a crucial factor in the analysis [15].

Consequently, there has been a resurgence of interest in the study of shadows of black holes and compact objects in diverse theories of gravity, in addition to the occurrence of black holes within Einstein's general relativity theory. This interest manifested in a variety of forms and from a range of perspectives, encompassing different spacetime dimensions. A number of studies have been conducted in a variety of directions, including the apparent shapes of black hole shadows in different black hole configurations and spacetime geometries [16-20], shadows of non-rotating and rotating black hole spacetimes has been demonstrated in different modified theories of gravity (22,33). Moreover, it has been demonstrated that the examination of shadows has the capacity to restrict the charge of a black hole (34). Furthermore, it have been found to exist within dynamically ever-evolving spacetimes [24]. In this study, the focus is twofold: firstly, photon capture in gravitationally intense stars: A study of Synge's black hole shadow. Secondly, on black hole backgrounds of the Reissner-Nordström-de Sitter type, which emerge from Einstein's General Relativity.

Material and methods

I. from Euclidean Geometry to Gravitational Distortion

A black hole is defined as a celestial object that captures all light falling onto it and emits no light. Consequently, even a rudimentary examination would suggest that an observer would perceive a dark spot in the sky where the black hole should be located. Nevertheless, as a consequence of the pronounced bending of light rays by the BH gravitational field, the size and shape of this area deviates from the predictions of naive Euclidean geometry for a non-gravitating black ball.

II. Gravitational Collapse

To date, the focus has been on "eternal black holes," defined as black holes that persist indefinitely within a static space-time continuum. However, the prevailing hypothesis is that the (stellar or supermassive) black holes that are directly observed in Nature have come into existence by gravitational collapse. It is evident that in such circumstances, the shadow would not persist indefinitely; rather, it would gradually materialize over time, even for an observer maintaining a constant distance from the black hole's center.

III. Exploring Photon Capture in Gravitationally Intense Stars

The term gravitationally intense is employed in the context of stars of a certain mass m and Schwarzschild radius r_0 to denote those objects for which the ratio $2m/r_0$ is only slightly less than unity. A narrow critical cone is formed as photons from the star escape, while others are drawn back. As this ratio approaches unity, the critical cone metamorphoses into a line, and all escaping photons emanate directly from the surface. So, Synge's explanation from 1965 of the "Black Hole Shadow" is all about these super powerful stars.

IV. Identifying Metrics in Spherically Symmetric Geometries

Firstly, the following general expression is employed for the calculation of the inclination angle of a light ray emitted from the observer into the past. This expression is understood to be general in the context that it will incorporate a constant of motion; the specific value of this constant remains undefined. Consequently, the expression is applicable to all emitted light rays. The radius coordinate of the point of closest approach is utilised as the relevant constant of motion in the subsequent analysis [29].

V. Plasma Influences Shadow

In order to take the effect of a medium on light propagation into account, numerical investigations, such as ray tracing, are typically employed. In addition, it is notable that there is a particular kind of medium whose effect on the shadow can be analysed. Indeed, this medium has been analysed for a number of space-times. The medium in question is a non-magnetised, pressure-less electron-ion plasma. In this section, an evaluation of the outcomes derived from the calculation of the impact of the aforementioned medium on the shadow is conducted. In this. The approach of the plasma can be determined by observing the influence of the plasma on the space-time trajectories of the rays.

Results and discussion

I. from Euclidean Geometry to Gravitational Distortion: The Complex Representation of Black Hole Shadows

In the specific case of spherically symmetric black holes, the discrepancy between the shadow and the imaginary Euclidean image of the black hole is manifest exclusively in terms of the angular size. The shadow is approximately two and a half times larger see (Figure 4). In the context of a rotating black hole, the shadow experiences an alteration in its configuration, manifesting as deformation and a lateral flattening. The dimensions and form of the shadow are determined not solely by the characteristics of the black hole under consideration, but also by the position of the observer. A plethora of words have been utilised throughout history to denote the visual manifestation of a black hole and associated concepts. Escape cone, see Synge [25], mirage around black hole, see Zakharov et al [26], photon ring, see Johannsen and Psaltis [27], Johnson et al [28] and black hole shadow, see Falcke, Melia and Agol [14].

II. Gravitational Collapse: A New Perspective on Black Hole Shadows

A substantial corpus of research exists, commencing with the foundational contributions of Ames and Thorne [34], which delineated the visual manifestation of a collapsing non-transparent star. Subsequent research can be found in the works of Jaffe [35], Lake and Roeder [36], and Frolov et al. [37]. In the aforementioned works, the predominant emphasis is placed on the frequency shift of light emanating from the surface of a collapsing star. Recent studies by Kong et al.[38, 39] and Ortiz et al.[40, 41] have examined the frequency shift of light passing through a collapsing transparent star. These studies thus contrast the collapse to a black hole with the collapse to a naked singularity. However, a paucity of literature has addressed the temporal evolution of the shadow's formation and the subsequent change in its angular radius, as perceived by an observer at a distance from the collapsing star. The latter question was the subject of investigation by Schneider and Perlick. [42]. A minimalist model was considered, comprising a non-transparent, spherically symmetric, collapsing sphere of dust, devoid of any luminosity at its exterior. In this case, the spacetime external to the ball is determined by the Schwarzschild metric, and the temporal behavior of the star's surface has been calculated in a classical paper by Oppenheimer and Snyder [43]. Schneider and Perlick's findings, based on these facts, suggest that the shadow's formation is a finite process. That is to say, after a finite time, an observer at a distance greater than the Schwarzschild radius would observe a circular shadow with an angular radius calculated using Synge's formula. This outcome could not have been readily predicted without the implementation of a calculation; in fact, based on intuition alone, a conjecture could have been made that the Synge formula would be approached only asymptotically.

III. Exploring Photon Capture in Gravitationally Intense Stars: A Study of Synge's Black Hole Shadow

In the context of spherical stars, under the assumption that $G = c = 1$, the radial coordinate r aligns with r_0. This finding suggests that the ratio $2m/r_0$ remains consistently less than unity. The term "gravitationally intense" is employ to denote instances in which this ratio approaches one. Dimensionless coordinates such as ρ and τ are introduced for theoretical convenience. Synge published the first detailed explanation of the "black hole shadow," using the standard Schwarzschild coordinates to derive his findings (i.e. $\rho \equiv r/2m$ and $\tau \equiv t/2m$)

$$ds^2 = 4m^2\{(1-\rho^{-1})d\rho^2 + \rho^2(d\theta^2 + sin^2\theta d\theta) - (1-\rho^{-1})dt^2\} \qquad (1)$$

It is evident from the evidence presented that, given the exclusive presence of m within the initial factor, light rays (null geodesics) emanating from a given stellar object can be addressed, to a certain extent, within a single argument. It can be demonstrated that, within the framework of a specified null geodesic, the selection of coordinates facilitates the establishment of a constant ϕ along the specified geodesic. In order to arrive at an optimal solution, it is necessary to calculate the second derivatives of both sides of the metric equation with respect to an affine parameter (ξ). The expression $\dot{s}^2 = (ds/d\xi) = 0$. signifies that the trajectory under consideration is that of light (null geodesics). In this instance, the symbol ξ is employed to denote the trajectory of light, specifically the null geodesics along which it moves. In this context, (s) denotes the spacetime interval, and (ξ) signifies an affine parameter.

$$(1-\rho^{-1})^{-1}\frac{d^2\rho}{d\xi^2} + \rho^2\frac{d^2\theta}{d\xi^2} - (1-\rho^{-1})\frac{d^2t}{d\xi^2} = 0 \qquad (2)$$

$$(1-\rho^{-1})^{-1}\dot{\rho}^2 + \rho^2\dot{\theta}^2 - (1-\rho^{-1})\dot{t}^2 = 0 \qquad (3)$$

The "two constants of motion", denoted by $\rho^2\dot{\theta} = \propto$ and $(1-\rho^{-1})\dot{t} = \alpha\beta$, refer to specific quantities that remain unchanged as light travels in the vicinity of a black hole. The aforementioned constants are related to the conservation of angular momentum and energy; therefore, as light moves in the gravitational field of the black hole, its angular momentum and energy remain constant. This theoretical framework facilitates comprehension of the trajectories that light can adopt when traversing the vicinity of a black hole. It has been determined that

$$F(\beta,\rho) = \frac{1}{\rho^4}\frac{d^2\rho}{d\theta^2} \qquad (4)$$

$$F(\alpha,\beta) = (\beta^2\rho^{-3} - (\rho-1)/\rho^3) \qquad (5)$$

Where β and ρ are representing a specified energy-related quantity for light rays, and, a dimensionless coordinate related to the distance from the black hole. It is imperative to comprehend the function's significance in elucidating the behaviour of light when interacting with a black hole's gravitational field. This function is particularly instrumental in determining the regions from which light may or may not escape. In the present investigation, the calculation of the inclination of the ray in question was undertaken with regard to the radial direction. This was achieved by employing the formula $cot\,\gamma = dr/rd\theta$. It is apparent that the Schwarzschild coordinate, denoted by r, possesses a geometrical significance. However, it is noteworthy that this significance bears no direct relationship to the measurement of the angle γ. It is imperative that the measurement of γ is conducted in accordance with the metric of the curved space t = const. That is

$$ds^2 = 4m^2[(1-\rho^{-1})^{-1}d\rho^2 + \rho^2(d\theta^2 + sin^2\theta d\phi^2)] \qquad (6)$$

$$cot\,\gamma = (1-\rho^{-1})^{-1/2}\rho^{-1}\frac{d\rho}{d\theta} \qquad (7)$$

$$cot^2\,\gamma = \rho^3(\rho-1)^{-1}\,F(\beta,\rho) = \frac{\rho^3\beta^2}{\rho-1} - 1 \qquad (8)$$

$$sin^2\,\gamma = (\rho-1)(\rho^{-3}\beta^{-2}) \qquad (9)$$

$$\beta^2 = (\rho-1)\rho^{-3}cosec^2\,\gamma \qquad (10)$$

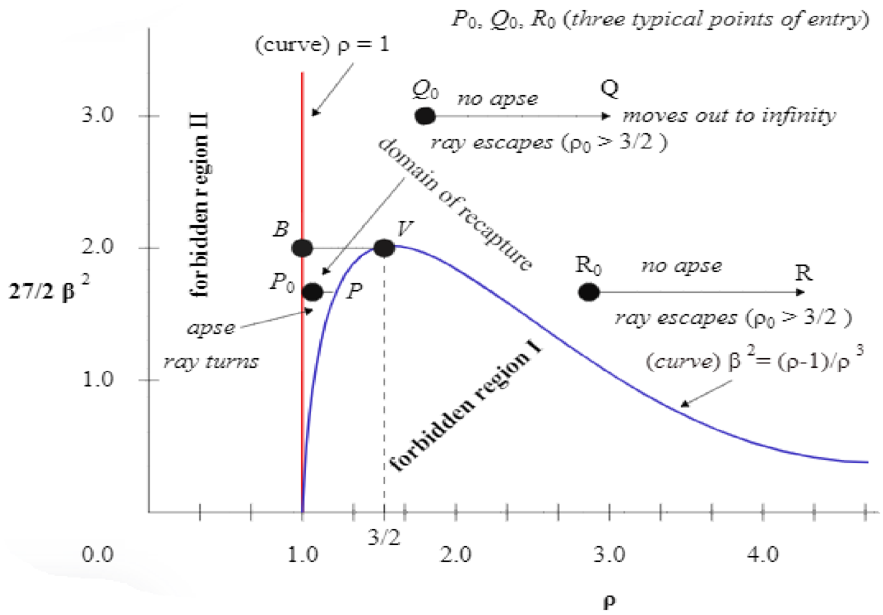

Figure: 1. Depiction of forbidden regions. If $\rho_0 > 3/2$ (i.e, if $r_0 > 3m$) all rays scape. The condition for escape is $\beta^2 > 4/27$.

From the conditions presented, it is evident that when the parameter ρ is set at $3/2$ as it depicted in (Figure 1), the maximum possible value for β^2 is equivalent to $4/27$. This is of consequence since it represents a substantial threshold, one which light rays originating from $\rho_0 > 3/2$ must surmount. The Synge equation is recovered by utilising observer coordinates as the initial parameters, in conjunction with these conditions.

$$(\rho_0 - 1)\rho_0^{-3} = \frac{4}{27} sin^2\chi \qquad (11)$$

Synge's equation was initially developed to calculate the escape of light rays from the surface of a "gravitationally intense star" when the angle satisfies $\gamma_0 < \chi$. However, it can also be applied to calculate the angular radius of the black hole shadow observed by a stationary observer at ρ_0. Furthermore, the hypothesis that the rays emitted by the object are deflected, yet not captured, necessitates a greater angular radius of the shadow. This is only possible if the rays are sent by the observer in such a way that the ratio of the deflected ray to the captured one satisfies the relation $\gamma_0 > \chi$. As shown in the figure 2. Where (red line) illustrates the calculation of the light ray emanating from the observer's position into the past, with an angle of incidence γ. It is imperative to note that the trajectory is calculated within the framework of Schwarzschild spacetime, with R denoting the radius coordinate at the point of closest approach. (blue line) shows the limiting light ray path. The definition of Synge's angle is also shown. light rays with a smaller angle of incidence would be absorbed by the event horizon. In contrast, those with a greater angle of incidence would be deflected to infinity, which is the illuminated region that the observer would perceive.

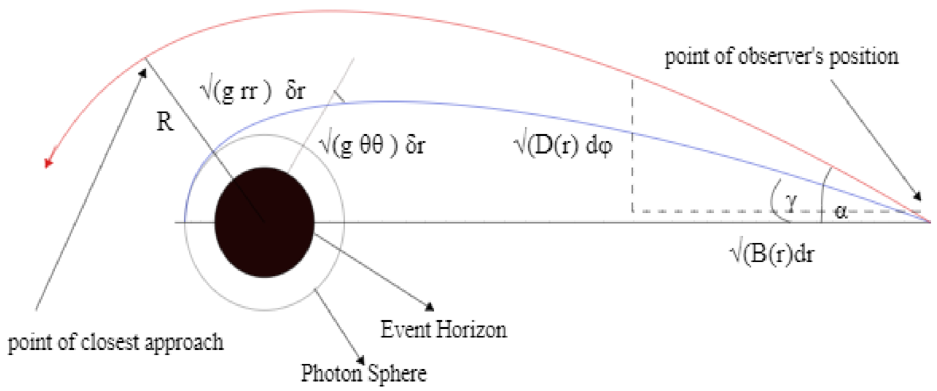

Figure: 2. Schematic representation of trajectories of light ray with a smaller angle of incidence would be absorbed by the event horizon. In contrast, those with a greater angle of incidence would be deflected to infinity.

IV. Identifying Metrics in Spherically Symmetric Geometries (Employing Lagrange Formalism)

Firstly, the following general expression is employed for the calculation of the inclination angle of a light ray emitted from the observer into the past. This expression is understood to be general in the context that it will incorporate a constant of motion; the specific value of this constant remains undefined. Consequently, the expression is applicable to all emitted light rays. The radius coordinate of the point of closest approach is utilised as the relevant constant of motion in the subsequent analysis [29]. In the second part of the investigation, it will be necessary to distinguish between the light rays which asymptotically approach unstable circular light orbits. The constant of motion must therefore be substituted into the general formula for the angle Eq. 22. In the event of multiple photon spheres, it is imperative to delineate the precise location of the light sources. In this study, the focus will be on objects that do not possess a photon sphere. It is noteworthy that, within the framework of spherically symmetric and static metrics, metrics with a photon sphere are precisely those for which an observer can theoretically perceive an infinite number of images of a light source (see Hasse and Perlick [30]). It is evident that a more efficacious and thoroughgoing approach to addressing Synge's problem is to be found in resolving the generalized versions of spherically symmetric geometries and employing the Lagrange formalism to calculate the constants of motion. The initial step in this methodological approach pertains to the identification of the metric of the problem under investigation. In the case at hand, the identified metric is as follows

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -A(r)c^2dt^2 + B(r)dr^2 + D(r)(d\theta^2 + sin^2\theta\phi^2) \qquad (12)$$

Or equivalently,

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + \mathrm{D}(r)(d\theta^2 + sin^2\theta\phi^2) \qquad (13)$$

$A(r)$, $B(r)$ and D(r) are all positive.

In order to obtain the conserved quantities, it is imperative to derive the associated Lagrangean. Specifically in the absence of electromagnetic interaction, the vacuum case can be expressed in the following manner with respect to the Lagrangian:

$$\mathcal{L}(x,\dot{x}) = \frac{1}{2}g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} \qquad (14)$$

For the geodesics takes the form

$$\mathcal{L}(x,\dot{x}) = \frac{1}{2}(-A(r)c^2\dot{t}^2 + B(r)\dot{r}^2 + D(r)(\dot{\theta}^2 + sin^2\theta\dot{\phi}^2) \qquad (15)$$

Due to the symmetry, it is sufficient to examine geodesics within the equatorial plane, i.e. $\vartheta = \pi/2, sin\,\vartheta = 1$. Subsequently, the t and ϕ components of the Euler-Lagrange equation are to be considered.

$$\frac{d}{d\xi}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^{\mu}}\right)\frac{\partial\mathcal{L}}{\partial x^{\mu}} = 0 \qquad (16)$$

Subsequently, the Euler-Lagrange equations stipulate that in the event of a coordinate not manifesting explicitly within the formulation of the action integral (i.e. $\partial_{x^{\mu}}\mathcal{L} = 0$), an associated constant of motion is present, which can be calculated as follows:

$$\lambda_{\mu} = \frac{\partial\mathcal{L}}{\partial\dot{x}^{\mu}} \qquad (17)$$

In the case of a spherically symmetric system,. These fields are responsible for establishing two constants of motion. Moreover, the condition $\mathcal{L} = 0$ for photons is also taken into account. So

$$-A(r)\dot{t} = -E \rightarrow \quad E = A(r)c^2\dot{t} \qquad (18)$$

$$D(r)\dot{\phi} = L_z \qquad (19)$$

In lieu of employing the r component of the Euler-Lagrange equation, it is more expedient, for the purposes at hand, to utilise a first integral of the geodesic equation, namely (for light) $g_{\mu\nu}dx^{\mu}dx^{\nu} = 0$. Eq. (14) [31]. The constant of motion is often redefined in such a manner that it is expressed as a new constant, designated 'b'. It is the 'impact parameter', and is defined by:

$$b \coloneqq \frac{L_z}{E} = \frac{D(r)\dot{\phi}}{A(r)\dot{t}} = \frac{D(r)d\phi}{A(r)dt} \qquad (20)$$

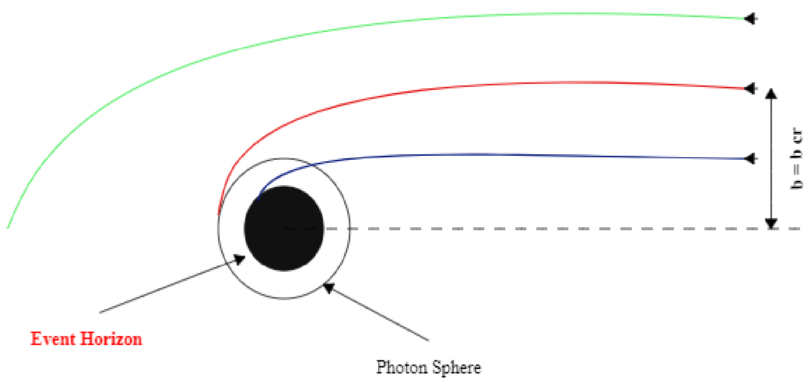

Figure: 3. the trajectory of a photon in a vacuum is determined by the impact parameter

The insertion of Eqs. (18 and 19) into Eq. $g_{\mu\nu}dx^{\mu}dx^{\nu} = 0$ results in a solution for $\frac{\dot{r}^2}{\dot{\phi}^2} = (dr/d\phi)^2$ .The subsequent application of this solution to the orbit equation for lightlike geodesics is a valid.

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{D(r)d\phi}{B(r)dt} = \left(\frac{D(r)}{A(r)}\frac{1}{b^2} - 1\right), \quad \frac{E^2}{{L_z}^2} = \frac{1}{b^2} \tag{21}$$

While this equation can be analytically integrated, as demonstrated in [32], which is the prevailing methodology for determining shadows from more intricate black holes, Synge's condition can also be expressed in terms of the metric components. At this stage, it proves advantageous to compute the turning points of these trajectories, which is tantamount to finding the forbidden regions from Synge's equation (4). These points can be found by:

$$\left.\frac{dr}{d\phi}\right|_{r_{ph}} = 0 \tag{22}$$

In the Schwarzschild case, the function $h(r)$ is equivalent to the effective potential which was introduced in equation (25, 58) of Misner et al. [33]. It can be demonstrated that there is a relationship between the impact parameter and the function $h(r)$ by means of the following equation:

$$b = h(R) \tag{23}$$

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{D(r)d\phi}{B(r)dt} = \left(\frac{h(r)^2}{h(R)^2} - 1\right) \tag{24}$$

(R) denotes the radius coordinate at the point of closest approach as can be seen from Figure.2. Which implies the solution

$$b^{-2} = \frac{A(r)}{D(r)} \tag{25}$$

It is imperative to acknowledge that alternative solutions, such as $D(r) = 0$ or $B(r) \rightarrow \infty$, are not viable options. This is due to the fact that these solutions imply that the metric is singular outside the event horizon. While the calculation of these objects' shadows is a conceivable undertaking, it does not represent the primary objective of the present study.

$$\cot\psi = \frac{\sqrt{B(r)}}{D(r)}\frac{dr}{d\phi}\Big|_R \tag{26}$$

The angle ψ between such a light ray and the radial direction. Now, it is useful to define the function $h(x)$ as

$$h^2(x) \equiv \frac{D(r)}{A(r)} \rightarrow b = h(R) \tag{27}$$

Using (26) together with (2), the angular radius from the shadow will be

$$cot^2\psi = \frac{h(r)}{h(R)} - 1 \tag{28}$$

This assertion is consistent with the findings outlined in [31], which demonstrate that this phenomenon persists at all points along the trajectory. In constructing the shadow, the following assumption is made: a static observer at radius coordinate r_0 sends light rays into the past. As is evident in (Figure.2). Synge's condition can be recovered through the application of trigonometry, thus enabling its expression in two forms.

$$sin^2\psi = \left(\frac{h(r_{ph})}{h(r_0)}\right)^2 \tag{29}$$

$$sin^2\psi = \frac{b_{cr}^2}{h(r_0)}, \quad \text{or} \quad sin^2\psi = \frac{A(r_0)b_{cr}^2}{D(r_0)} \tag{30}$$

In the present context, it is understood that $b_{cr} \equiv h(r_{ph})$ and r_{ph} the radial coordinate of circular photons orbiting the central mass Black Hole is indicated (i.e. the photospheres). As is well-known, these can be found by means of the Synge derivation. Firstly, it is necessary to identify from (21) the term, which, if employed, would result in the RHS being negative. In the context of spherically symmetric geometries, this would be

$$S(r;b) = \frac{D(r)}{A(r)}\frac{1}{b^2} - 1 \tag{31}$$

It is evident that $S(r;b)$ is formally equivalent to the function employed in Synge's derivation, with its scope of application being confined to the variable r. In order to ascertain the permissible domains for r, it is imperative that the condition $F(r;b) > 0$ is satisfied. The following procedure will allow the condition to be visualized

$$\frac{1}{b^2} = \frac{A(r)}{D(r)} \tag{32}$$

The aforementioned curve is thus pivotal in defining the turning points of photons, i.e. it sets $dr/d\phi = 0$. Photons that lie above the curve (32) are permitted, whilst those that lie below the curve are prohibited. It is noteworthy that the photon orbits are located at the extremities of this curve, exhibiting stability at minima and instability at maxima. The application of this condition is as follows:

$$\frac{d}{dr}\left(\frac{A(r)}{D(r)}\right)\Big|_{r_{ph}} = \frac{d}{dr}\frac{1}{b^2}\Big|_{r_{ph}} = 0 \tag{33}$$

The present method facilitates remarkably efficient calculation of shadows from spherically-symmetric black holes Furthermore, it is noteworthy that the derivation of this equation renders the equation for shadows of spherically symmetric metrics unchanged under the action of a conformal transformation (i.e. $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = f(r)g_{\mu\nu}$), where f(r) is an arbitrary function of the radial coordinate alone. This transformation implies that each metric component from (13) will be multiplied by this general function. It can then be demonstrated that f(r) cancels out from (30) and (33), thereby ensuring that neither r_{ph} nor the shadow will change.

A. Shadow Calculations in Schwarzschild Space-Time: A Detailed Analysis

For the Schwarzschild space-time

$$ds^2 = -\left(1 - \frac{2m}{r}\right)c^2dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\theta^2 + sin^2\theta d\phi^2), \qquad m = \frac{Gm}{c^2}$$

Or equivalently from Eq. (1) that

$$A(r) = 1 - r^{-1} = \frac{1}{B(r)}, \qquad D(r) = r^2$$

Photosphere r_{ph} can be found from Eq. (33)

$$\frac{d}{dr}\left(\frac{1 - r^{-1}}{r^2}\right)\Big|_{r_{ph}} = 0 \quad \to phtonshere\ (r_{ph)} = 3/2$$

Expression (11) can now be recovered using (30).

..

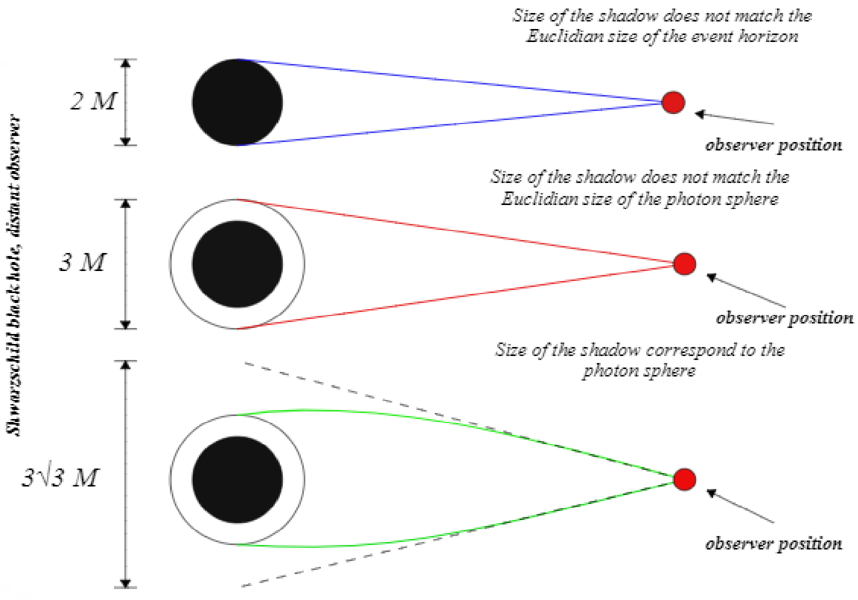

Figure.4. Compare the size of the black hole (event horizon), the photon sphere and the shadow's angular size.

B. Charged Black Holes Explained: The Reissner-Nordström Metric

The Reissner-Nordström (RN) solution is a theoretical framework that describes the behaviour of charged black holes. The metric components of this solution are as follows:

$$A(r) = B(r)^{-1} = , \quad 1 - \frac{2m}{r} + \frac{q^2}{r^2}, \qquad D(r) = r^2$$

Or equivalently

$$A(r) = B(r)^{-1} = 1 - r^{-1} + \beta^2 r^{-2}$$

$$(2\mathrm{M})(\beta) \equiv \sqrt{q_c^2 + q_m^2}$$

The term "M," representing the mass of the black hole, is introduced. This is necessary for ensuring dimensional invariance, with "β" assumed to be dimensionless. The electric charge is denoted by "q_c," while "q_m" denotes the magnetic charge. The quadratic dependence on β is indicative of the charge invariance of the metric. This outcome is anticipated, given that the negative and positive charges exert an equivalent influence on the energy (i.e. the mass) of the black hole. Using (33).

$$\frac{d}{dr}\left(\frac{1 - r^{-1} + \beta^2 r^{-2}}{r^2}\right)\Bigg|_{r_{ph}} = 0$$

r_{ph} can be determined as

$$r_{ph_\pm} = \frac{3}{4}\left(1 \pm \sqrt{1 - \frac{32}{9}\beta^2}\right) \tag{34}$$

The aforementioned equation provides an upper limit to the charge of the black hole, which is expressed as $\beta^2 > 1/4$. Evidence has been presented which indicates that $r_{ph_+} > r_{h_+} > r_{ph_-} > r_{h_-}$ for any physically acceptable value of β. This finding suggests that only the r_{ph_+} photosphere coordinate will be relevant for the shadow. Once more, the r_{ph} can be calculated by means of Eq. (33).

$$sin^2\psi = \frac{1}{32}\, \rho_0^{-4} \left\{\frac{(\beta^2)(3+\sqrt{9-32\beta^2})^4}{(3-8\beta^2+\sqrt{9-32\beta^2})}\right\} \tag{35}$$

It can be deduced that when $\beta = 0$, the Schwarzschild solution is retrieved.

C. Shadows in Kottler Space-Time: Perspectives from a Static Observer

From the Kottler space-time

$$A(r) = B(r)^{-1} = 1 - 2mr^{-1} - \frac{1}{3}\Lambda r^2 \quad (36)$$

In this section, the focus is directed towards the examination of charged black holes in universes that possess a non-vanishing cosmological constant. Well, in the first case, the metric's going to be made up of these components.

$$A(r) = B(r)^{-1} = 1 - r^{-1} + \beta^2 r^{-2} - \Lambda r^2, \quad D(r) = r^2 \quad (37)$$

In this text, the symbol "$\Lambda \equiv \frac{1}{3}(2M)^2 \tilde{\Lambda}$" is employed to denote the dimensionless cosmological constant. The number of horizons is contingent on the values of the pair of parameters $\{\beta, \Lambda\}$. The ensuing discussion will focus on the three distinct solutions that are delineated (Figure. 5). In order to facilitate a comprehensive analysis, it is imperative to undertake a systematic examination of these three solutions in the context of the critical values of the parameters in question.

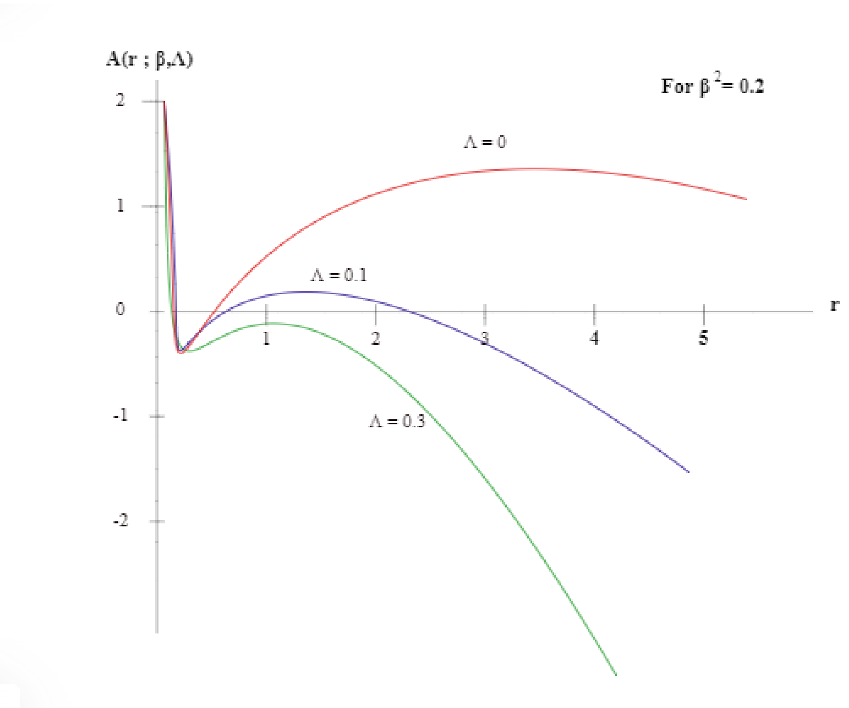

Figure: 5. A comparison of three solutions in terms of the critical values of the parameters. The number of horizons depends on {β, Λ}.

For a positive value $\Lambda > 0$ of the cosmological constant, a cosmological horizon. r_C, is formed. However, for a negative value $\Lambda \leq 0$ of the cosmological constant, no cosmological horizon is present. Furthermore, it is noteworthy to mention that under specific parameter values, these solutions exhibit the inner and outer horizons of the charged black hole (Reissner- Nordstrom solution). $A(r)r^2 = 0$, to vanish. This can be further elucidated by

$$\Lambda(256\beta^6\Lambda^2 + (27 + 128\beta^4 - 144\beta^2)\Lambda - 4 + 16\beta^2) = 0 \quad (38)$$

Therefore

$$\Lambda_c^{\pm} = \frac{1}{512}\beta^{-6}\left\{-27 + 16\beta^2(9 - 8\beta^2) \pm \sqrt{(9 - 32\beta^2)^3}\right\} \quad (39)$$

It can therefore be concluded that, subsequent to the initial phase, BHs will exist solely within the interval

$$\Lambda_c^{-}(\beta^2) < \Lambda < \Lambda_c^{+}(\beta^2). \quad (40)$$

The different regions are clearly illustrated in (Figure.6).

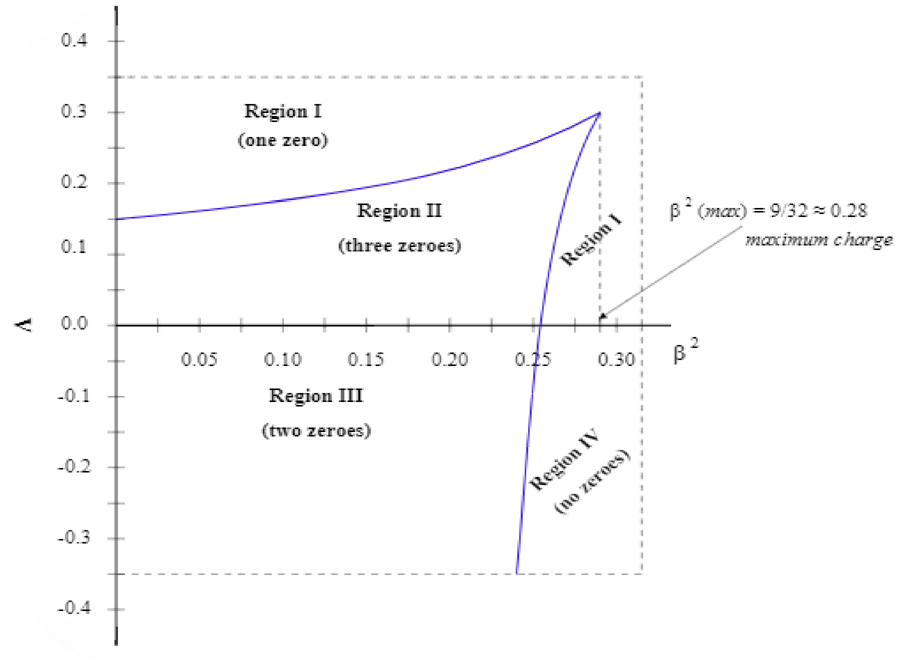

Figure: 6. Illustrates (Λ, β) diagram. Region I (one solution), Region II (three solution), Region III (two solutions) and Region IV (no solution)

Black hole (34) due to the quadratic dependence of the cosmological constant term on r. It can be demonstrated that r_{ph_-} will invariably be situated within the event horizon, whilst r_{ph_+} will be positioned between the event horizon and the cosmological horizon. It is evident that a direct correlation between the radial coordinate of the photosphere and that of the Reissner-Nordström cosmological horizon is to be expected. It is now possible to obtain the expression of the shadow for Reissner – Nordstrom - Kottler Black hole. by employing r_{ph_+} as opposed to r_{ph_-}.

$$sin^2\psi = \frac{Q(\beta^2)^4\, N(\beta^2;\Lambda,\rho_0)}{8\beta^2(D(\beta^2)\Lambda-4)-128\beta^4\Lambda-Q(\beta^2)(27-4\Lambda)} \quad (41)$$

It has been demonstrated that solutions for Reissner-Nordström and Schwarzschild are recoverable through the limits $\Lambda \to 0$ and $\Lambda, \beta \to 0$, respectively. It is intriguing to observe that positive values of Λ permit for Black Hole can exceed the prescribed threshold of $\beta^2 > 1/4$. For negative and small values of $|\Lambda|$, the Reissner-Nordström condition is recovered for $\beta^2 < 1/4$, and for more negative values of Λ, the maximum permitted value for β^2 gets monotonically smaller.

V. Plasma Influences Shadow Formation in Space-Time metric

It is evident that all plasma effects which necessitate a description in terms of wave optics are thus disregarded, as is the phenomenon of absorption. In a plasma, the rays are no longer lightlike geodesics of the space-time metric; rather, they are influenced by the plasma frequency. For a spherically symmetric a static metric

$$g_{\mathrm{ik}}dx^i dx^k = -A(r)dt^2 + B(r)dr^2 + D(r)(d\theta^2 + sin^2\theta\phi^2)$$

The values $A(r)$, $B(r)$ and $D(r)$ are considered positive. It is hypothesised that the spacetime is populated by a static, non-magnetised, cold, and inhomogeneous plasma whose electron plasma frequency ω_{ep} is a function only of the radius coordinate,

$$\omega_{ep}{}^2 = 4\pi e^2(\frac{1}{m_e})N(r) \quad (42)$$

Where, m_e and $N(r)$ denote the charge of the electron, the mass of the electron, and is the number density of the electrons, respectively. A distinctive characteristic of a non-magnetised, pressure-less electron-ion plasma is that the rays are influenced by the plasma in only one way: the plasma frequency, i.e. the electron density. However, this is no longer the case when magnetic fields or pressures are taken into account, or when electron-positron plasmas are considered, in which case the positron density also becomes a relevant factor.

The Hamiltonian formalism is utilised, enabling the allocation of a momentum co-vector (or wave co-vector) to each ray. The temporal component of these rays, as perceived by an observer field, can be interpreted as a frequency. It can thus be deduced that, in an arbitrary space-time, light rays constitute solutions to Hamilton's equations.

$$\dot{p}_p = -\frac{\partial\mathcal{H}(x,p)}{\partial x^i}, \qquad \dot{x}^i = \frac{\partial\mathcal{H}(x,p)}{\partial p_p}, \qquad \mathcal{H}(x,p) = 0 \qquad (43)$$

$$\mathcal{H}(x^i,p^i) = \frac{1}{2}\left[g^{ik}(r)p_i p_k) + (\omega_{ep}(r)^2\right] = 0 \qquad (44)$$

The initial rigorous derivation of this result from Maxwell's equations on a general-relativistic space-time with the plasma modelled in terms of two charged fluids was provided by Breuer and Ehlers [44]. In fact, Breuer and Ehlers even contemplated the broader scenario of a magnetised plasma. It is also noteworthy that the derivation made use of the fact that ions are much more inert than electrons; consequently, this derivation does not apply to an electron-positron plasma. As demonstrated in the Hamiltonian form, it can be deduced that the light rays in the plasma constitute timelike geodesics of a conformally rescaled metric. It is possible to associate a frequency, $\omega(r) = -U^i(r)p_i$ with each light ray by means of the four-velocity $U^i(r)$ of an observer field. This facilitates the characterisation of the propagation of light by means of an index of refraction.

$$n^2 = -\frac{\omega_{ep}}{|\omega(r)^2|^2} + 1 \qquad (45)$$

The distinctive property of a plasma that underpins its classification as a dispersive medium is its frequency-dependent optical index, a characteristic which distinguishes it from other media. Synge previously provided a comprehensive discussion on light propagation in a medium with a frequency-dependent index of refraction [45]. Synge did not specifically consider plasma as a media of this class, although it does belong to it as a special case.

Within the confines of a spherically symmetric and static metric (i.e. a metric of the form outlined in Eq. (12), the light rays traversing a plasma characterised by a given plasma frequency, denoted by $\omega(r)$, can be determined through analytical integration. The impact of plasma on the shadow was analysed by Perlick, Tsupko and Bisnovatyi-Kogan [46] under the assumptions of both the space-time and the plasma being spherically symmetric and static. The prevailing assumption was that the space-time continuum was asymptotically flat, and that the observer's position lay between infinity and an unstable photon sphere. Subsequently, the angular radius, ψ, of the shadow was determined using the following equation: [46].

$$sin^2\psi = \frac{h(r_{ph})^2}{h(r_0)^2} \qquad (46)$$

So that

$$h(r)^2 = \frac{D(r)}{A(r)}\left(-A(r)\frac{\omega_{ep}(r)^2}{\omega_0^2} + 1\right) \qquad (47)$$

$$h(r) = r\left[A(r)^{-1} - \frac{\omega_{ep}(r)^2}{\omega_0^2}\right] \to h(r) = \sqrt{\frac{r}{r-2M} - \frac{\omega_{ep}(r)^2}{\omega_0^2}},$$

$$A(r) = 1 - \frac{2M}{r}$$

It is posited that this function generalises in a natural way to the plasma case. The corresponding function $h(r)^2 = \frac{D(r)}{A(r)}$ is thus generalised. In this model, the quantity ω_0 denotes the frequency of light at infinity as perceived by a static observer, while r_0 represents the observer's radius coordinate and r_{ph} signifies the radius coordinate of the photon sphere. It is imperative to note that the constant of motion, designated here by the symbol ω_0, must not be erroneously equated with the photon frequency, designated by $\omega(r)$, as measured by a static observer at position r. These values are interconnected by the gravitational redshift relation, which is expressed as follows:

$$\omega(r) = \frac{C}{\omega_0}\left(\frac{1}{\sqrt{-g_{tt}(r)}}\right) \quad (48)$$

The aforementioned has been employed in the context of an asymptotically flat space-time, specifically the g_{tt} (Generalized Unitary Transformation) $\rightarrow -c^2$ at infinity. This is a crucial consideration. It is imperative to employ Synge's approach [45] with the refractive index (45) when utilising this method. Subsequent to consideration of equation (48), it can be expressed that the refractive index can be expressed as a function of r and ω_0, for a plasma frequency $\omega_{ep}(r)$ on a spherically symmetric and static space-time (12).

$$n^2 = \left(-A(r)\frac{\omega_{ep}(r)^2}{\omega_0^2}\right) + 1 \quad (49)$$

Two additional remarks are offered at the end of this section. Firstly, we discuss more general plasma models. As per the work of Broderick and Blandford [48] and [49], a significant stride was made in figuring out the size of shadows in a magnetised plasma. In particular, they calculated the photon capture cross-sections. They considered a number of different plasma densities and magnetic field strengths. Lately, the results of studying shadows of black holes surrounded by a plasma have been made to include all cases of a clear, spread-out medium with a given refractive index [50]. In more detail, the shadow has been calculated using analytical methods. These calculations were based on a general spherically symmetric metric in the presence of a medium with a spherically symmetric refractive index. Various plasma models can be studied using these results, extending beyond the cold plasma case.

Secondly, we note that, in a recent paper, the Event Horizon Telescope (EHT) Collaboration reported that their observations of M87 are consistent with general relativistic magneto hydrodynamic models of magnetically arrested accretion disks. Bisnovatyi-Kogan and Ruzmaikin first suggested this model of an accretion flow in a large-scale magnetic field [51,52], which was then investigated further by Igumenshchev et al. [53] and Narayan et al. [54].

Conclusion

A thorough analysis of the shadows cast by spherically-symmetric black holes is provided in this paper. It establishes a general approach for calculating shadows in spherically symmetric geometries, the application of this approach being used to investigate the properties of the most general stationary and spherically symmetric black holes. Such as with the Reissner-Nordström (anti-) dSitter metric. This study employs a methodology that illustrates the conformally invariance of the shadow associated with any spherically symmetric solution. The upper and lower limits on the parameters β and λ for the pair are presented and discussed in (section B). The determination of the maximum charge that can be attained by a black hole has the potential to yield the value of the cosmological constant. However, given the relatively negligible values of this constant as predicted by contemporary theories, its impact on the results is considered negligible. It appears that, to the best of our current understanding, the issue of deriving shadows for the Reissner-Nordström-Kottler metric has not yet been explored within the extant body of literature on the subject. Subsequent research endeavours may involve the exploration of the properties that are inherent in shadows of other Black Hole types, such as axially symmetric or time-dependent mass configurations. It is possible to derive the intrinsic characteristics of an object by subjecting it to close scrutiny. It is evident that the analytical calculation of the shadow, for a considerable range of metrics, has now been accomplished. The calculation of the angular size of the shadow, i.e. the area of its silhouette, in any spherically symmetric and static metric can be performed by means of the formula (29), for any position r_0 of a static observer. The radius of the photon sphere in this metric is determined by the formula (33). In the context of an asymptotically flat metric, the linear size of the shadow at large distances is determined by the critical impact parameter, which can be calculated using the formula $b_{cr} = G(r_{ph})$.

It is customary to depict the boundary curve of the shadow in terms of impact parameters; however, it is imperative to acknowledge that empirical observations refer to measurements of angles in the observer's sky. In the context of asymptotically flat space-time, the central region can be considered to be within the safe zone for observers situated at a considerable distance from the centre. This assertion is supported by the empirical evidence which demonstrates that, when dividing the impact parameters by the radial coordinate of the observer, the corresponding angles are obtained. However, in scenarios where the space-time is not asymptotically flat, such as the presence of a cosmological constant, conventional calculation methods for critical impact parameters are inadequate in elucidating the size and shape of the shadow. As has been emphasised on multiple occasions throughout this text, the construction of the shadow is contingent on the photon region, as opposed to the existence of a horizon. Consequently, the analytical methods employed to determine the shadow are not exclusive to black holes but extend to other objects that possess sufficient compactness to exhibit a photon region. Such objects are frequently designated as "ultracompact"; for instance, wormholes are considered to be a subset of this category. In the context of spherically symmetric and static spacetimes, it is possible to calculate the influence of a cold, non-magnetised plasma on the shadow by employing formula (46), on the assumption that there exists a unique unstable photon sphere at radius r_{ph}. In axisymmetric and static space-times, the plasma frequency (and, consequently, the electron density of the plasma) must be specified in a particular form to enable a Carter constant, which is indispensable for the analytical calculation of the shadow. It is strongly advocated that further advancement is made in the development of analytical methods for calculating the shadow. This would serve to complement the ongoing numerical studies, which are of high relevance for the purpose of comparison with observations. In the context of future research projects, there is a need for a more detailed study of the shadow, with particular reference to the photon region, in rotating wormhole space-times. In addition, the influence of a magnetised plasma on the shadow must be considered.

References

[1] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett.116, no. 6, 061102 (2016).
[2] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett.116, no. 24, 241103 (2016).
[3] R. Schodel et al., Nature 419, 694 (2002).
[4] A. M. Ghez et al., Astrophys. J. 689, 1044 (2008).
[5] J. M. Bardeen, "Timelike and null geodesics in the kerr metric". Proceedings of the Ecole d'Et de Physique Thorique: Les Astres Occlus: Les Houches 1972, p. 215-240 (1973).
[6] J.-P. Luminet, Astron. [Evet Horizon Telescope Collaboration]. Astrophys. 75, 228 (1979).
[7] K. Akiyama et al. Astrophys. [Evet Horizon Telescope Collaboration]. J. 875, no. 1, L1 (2019).
[8] K. Akiyama et al. [Evet Horizon Telescope Collaboration]. Astrophys. J. 875, no. 1, L2 (2019).
[9] K. Akiyama et al. [Evet Horizon Telescope Collaboration]. Astrophys. J. 875, no. 1, L3 (2019).
[10] K. Akiyama et al. [Evet Horizon Telescope Collaboration]. Astrophys. J. 875, no. 1, L4 (2019).
[11] K. Akiyama et al. [Evet Horizon Telescope Collaboration]. Astrophys. J. 875, no. 1, L5 (2019).
[12] K. Akiyama et al. [Evet Horizon Telescope Collaboration]. Astrophys. J. 875, no. 1, L6 (2019).
[13] J. L. Synge, Mon. Not. Roy. Astron. Soc. 131, no. 3, 463 (1966).
[14] H. Falcke, F. Melia and E. Agol, Astrophys. J. 528, L13 (2000).
[15] K. Hioki and K. i. Maeda, Phys. Rev. D 80, 024042 (2009).
[16] C. Bambi and K. Freese, Phys. Rev. D 79, 043002 (2009).
[17] K. Hioki and U. Miyamoto, Phys. Rev. D 78, 044007 (2008).
[18] K. S. Virbhadra, Phys. Rev. D 79, 083004 (2009).
[19] T. Johannsen, Astrophys. J. 777, 170 (2013)
[20] V. Bozza, F. De Luca and G. Scarpetta, Phys. Rev. D 74, 063001 (2006)
[21] R. A. Konoplya and A. F. Zinhailo, arXiv: 2003.01188 [gr-qc].
[22] L. Amarilla and E. F. Eiroa, Phys. Rev. D 87, no. 4, 044057 (2013).
[23] F. Atamurotov, A. Abdujabbarov and B. Ahmedov, Phys. Rev. D 88, no. 6, 064004 (2013).
[24] A. K. Mishra, S. Chakraborty and S. Sarkar, Phys. Rev. D 99, no. 10, 104080 (2019).
[25] J. L. Synge, Mon. Not. Roy. Astron. Soc. 131 (1966) 463
[26] A.F. Zakharov, A.A. Nucita, F. DePaolis, G. Ingrosso, New Astronomy 10 (2005) 479
[27] T. Johannsen, D. Psaltis, Astrophys. J. 718 (2010) 446
[28] M.D. Johnson, A. Lupsasca, A. Strominger, et al, Science Advances 6 (2020) eaaz1310
[29] "Unpublished" 2019, p. 123
[30] W. Hasse and V. Perlick, Gravitational lensing in spherically symmetric static spacetimes with centrifugal force reversal, Gen. Rel. Grav. 34 (2002) 415
[31] V. Perlick and O. Y. Tsupko, Physics Reports 947, 1 (2022), ISSN 0370-1573, calculating black hole shadows: Review of analytical studies.
[32] E. Hackmann and C. L¨ammerzahl, Phys. Rev. D 78, 024035 (2008).
[33] C.W. Misner, K.S. Thorne, J.A. Wheeler, Gravitation, Freeman, San Francisco (1973)
[34] W.L. Ames, K.S. Thorne, The optical appearance of a star that is collapsing through its gravitational radius, Astrophys. J. 151 (1968) 659
[35] J. Jaffe, Collapsing objects and the backward emission of light, Ann. Phys. (NY) 55 (1969) 374
[36] K. Lake, R.C. Roeder, Note on the optical appearance of a star collapsing through its gravitational radius, Astrophys. J. 232 (1979) 277
[37] V. P. Frolov, K. Kim, H. K. Lee, Spectral broadening of radiation from relativistic collapsing objects, Phys. Rev. D 75 (2007) 087501

[38] L. Kong, D. Malafarina, C. Bambi, Can we observationally test the weak cosmic censorship conjecture? Eur. Phys. J. C 74 (2014) 2983
[39] L. Kong, D. Malafarina, C. Bambi, Gravitational blueshift from a collapsing object, Phys. Lett. B 741 82 (2015)
[40] N. Ortiz, O. Sarbach, T. Zannias, Shadow of a naked singularity, Phys. Rev.D 92 (2015) 044035
[41] N. Ortiz, O. Sarbach, T. Zannias, Observational distinction between black holes and naked singularities: the role of the redshift function, Class. Quantum Grav. 32 (2015) 247001 [42] S. Schneider, V. Perlick, Gen. Rel. Grav. 50 (2018) 58
[43] J. R. Oppenheimer, H. Snyder, On continued gravitational contraction, Phys. Rev. 56 (1939) 455
[44] R. Breuer, J. Ehlers, I, Proc. Roy. Soc. London. A 370 (1980) 389
[45] J.L. Synge, Relativity: The General Theory, North-Holland, Amsterdam (1960)
[46] V. Perlick, O. Yu. Tsupko, G. S. Bisnovatyi-Kogan, Influence of a plasma on the shadow of a spherically symmetric black hole, Phys. Rev. D 92 (2015) 104031
[47] V. Perlick, O. Yu. Tsupko, Physics Reports. Rev. D 95 (2017) 104003
[48] A. Broderick, R. Blandford, Covariant magnetoionic theory – I. Ray propagation, Mon. Not. Roy. Astron. Soc. 342 (2003) 1280
[49] A. Broderick, R. Blandford, Covariant magnetoionic theory – II. Radiative transfer, Mon. Not. Roy. Astron. Soc. 349 (2004) 994
[50] O.Yu. Tsupko, Deflection of light rays by a spherically symmetric black hole in a dispersive medium, Phys. Rev. D 103 (2021) 104019
[51] G. S. Bisnovatyi-Kogan, A. A. Ruzmaikin, The accretion of matter by a collapsing star in the presence of a magnetic field, Astrophys. Space Sci. 28 (1974) 45
[52] G. S. Bisnovatyi-Kogan, A. A. Ruzmaikin, The accretion of matter by a collapsing star in the presence of a magnetic field. II. Selfconsistent stationary picture, Astrophys. Space Sci. 42 (1976) 401
[53] I.V. Igumenshchev, R. Narayan, M.A. Abramowicz, Three-dimensional magnetohydrodynamic simulations of radiatively inefficient accretion flows, Astrophys. J. 592 (2003) 1042
[54] R. Narayan, I.V. Igumenshchev, M.A. Abramowicz, Magnetically Arrested Disk: an energetically efficient accretion flow, Publ. Astron. Soc. Japan 55 (2003) L69