AJAPAS

African Journal of Advanced Pure and Applied Sciences (AJAPAS)

Online ISSN: 2957-644X

Volume 4, Issue 3, 2025

Page No: 500-510

Website: https://aaasjournals.com/index.php/ajapas/index

ISI 2025: 1.126 SJIFactor 2024: 6.752 معامل التأثير العربي: 1.62

A novel Class of Lomax Parameters Estimation Using Bayesian and Non-Bayesian Approach

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فئة جديدة لتقدير معاملات توزيع لومكس باستخدام المنهج البايزي وغير البايزي

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Received: July 15, 2025	Accepted: September 05, 2025	Published: September 16, 2025

Abstract:

A new class of Lomax distribution has been introduced using DUS method, called the L-Lomax distribution. Mathematical formulations with some properties of L-Lomax distribution are provided. Maximum likelihood estimators and Bayesian estimators, corresponding to two informative priors gamma and exponential based on complete and incomplete data, are obtained. Monte Carlo simulations for different L-Lomax generated random samples ranging from 500 to 50000 were performed. Additionally, various informative priors such as exponential, Weibull, gamma, beta and chi-square are used to illustrate the flexibility of the proposed distribution among other competitive models.

Keywords: Bayesian approach, DUS method, Lomax distribution, Monte Carlo, Risk function.

الملخص

تم في هذه المقالة اقتراح فئة جديدة من توزيعات لوماكس بالاعتماد على طريقة DUS وأطلق عليه توزيعL-Lomax . تم اشتقاق عدد من الخصائص الرياضية الأساسية لهذا التوزيع, كما تم الحصول على تقديرات معلماته باستخدام طريقة الامكان الأعظم والاستدلال البايزي بالاعتماد على بيانات كاملة وأخرى غير مكتملة. ولتقييم أداء الطرائق التقديرية، نُفذت تجارب محاكاة مونت كارلو باستخدام عينات عشوائية مولدة من توزيع المقترحL-Lomax بأحجام تراوحت بين (500-50000) مفردة. إضافة إلى ذلك، تم استخدام توزيعات قبلية متنوعة مثل الأسّي، وويبل، وجاما، وبيتا، ومربع كاي، بغرض إبراز مرونة التوزيع المقترح بين النماذج التنافسية الأخرى.

الكلمات المفتاحية: النهج البايزي، طريقة DUS، توزيع لومكس، مونت كارلو، دالة المخاطرة.

Introduction

Lomax distribution has attracted the attention of researchers over the years due to its ability to model heavy-tailed data and its applicability in various fields such as economics, actuarial science, and engineering [1]. Recent studies have expanded its use in survival analysis, financial risk modeling, and reliability engineering [10]. This distribution, also called Pareto type-II or Pearson type-VI distribution, provides a flexible framework for analyzing failure data and life tests, making it a popular choice for statisticians and practitioners. It is also referred to as the Pareto type-II or Pearson type-VI distribution, which was first introduced by [7] in his analysis of failure data. Its probability density function (pdf) with shape parameter $\alpha > 0$and scale parameter $\beta > 0$ is

$$f(t;\alpha,\beta)=\frac{\alpha}{\beta}\left(1+\frac{t}{\beta}\right)^{-(\alpha+1)} \quad ; \quad t>0 \quad \alpha,\beta>0, \tag{1}$$

with cumulative distribution function (cdf)

$$F(t;\alpha,\beta)=1-\left(1+\frac{t}{\beta}\right)^{-\alpha} \quad ; \quad t>0 \quad \alpha,\beta>0 \tag{2}$$

Many authors have been interested in studying the properties of Lomax distribution and estimating its unknown parameters, for example [8] estimated the shape parameter, reliability function, and failure rate of Lomax distribution using Bayesian method, [4] discussed the Bayesian estimates of Lomax model parameters based on censored type-II samples. [6] used Bayesian method and Bayesian predictive method to estimate the unknown shape parameter and some lifetime characteristics such as reliability function and risk function of Lomax distribution based on censored type-II samples [5]. Estimated the shape parameter and the failure rate of the Lomax distribution assuming that the measurement parameter is known using censored type-II samples and use of gamma distribution as prior distribution.

Recently, innovative methods for parameter estimation in Lomax distribution, such as the Markov Chain Monte Carlo (MCMC) approach and the application of machine learning models, have been explored [2]. These advancements highlight the continuous interest in enhancing the estimation techniques for this distribution, especially under varying sample sizes and censoring schemes. Furthermore, hybrid methods combining Bayesian estimation with computational techniques like importance sampling are gaining momentum [9]. The application of Lomax distribution in modern reliability frameworks, including IoT-based systems and AI-driven predictive maintenance, further demonstrates its relevance in contemporary research (Zhou et al., 2023).

In this article, the DUS method, a versatile approach proposed by [3], is utilized to introduce a novel class of the Lomax distribution, termed the L-Lomax distribution and abbreviated as $\mathrm{LLO}(\alpha,\beta)$. The DUS method is particularly known for its flexibility and effectiveness in generating extended forms of existing distributions, enabling enhanced adaptability in diverse statistical modeling scenarios. In the rest of this paper, the L-Lomax distribution and its associated properties are elaborated in Section 2. Maximum likelihood estimation methods, applied to both complete and incomplete data, are detailed in Section 3. Bayesian estimation techniques, similarly based on complete and incomplete data, are discussed in Section 4. Finally, Section 5 presents Monte Carlo simulations conducted using L-Lomax-generated data to demonstrate the flexibility and applicability of the proposed distribution. These simulations underscore the practical utility of the L-Lomax distribution in addressing challenges in reliability and survival analysis.

Material and methods

L-Lomax Distribution with Some Related Properties : Let f(t) and F(t) are a pdf and a cdf of any base distribution. According to the DUS method with $t>0$; $\alpha,\beta>0$ the pdf can be defined as :

$$g(t)=(e-1)^{-1}f(t) \quad e^{F(t)} \tag{3}$$

By substituting Equations (1) and (2) in equation (3), the new pdf class of L-Lomax distribution with $t>0$; $\alpha,\beta>0$ is :

$$g(t)=\frac{\alpha}{(e-1)\ \beta}\left(1+\frac{t}{\beta}\right)^{-(\alpha+1)} e^{1-\left(1+\frac{t}{\beta}\right)^{-\alpha}},$$

with cdf,

$$G(t)=(1-e)^{-1}\left(1-e^{1-\left(1+\frac{t}{\beta}\right)^{-\alpha}}\right).$$

The pdf of L-Lomax distribution takes the form of the letter *L*, and then called the L-Lomax distribution, see Figure 1. The cdf is increasing, and directly related with the shape parameter α when the scale parameter β is fixed, and inversely related with the scale parameter β when shape parameter α is fixed, see Figure 2.

Row moments of L-Lomax distribution is:

$$\mu'_r=\frac{\beta^r e}{(e-1)}\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\ \Gamma\left(1-\frac{k}{\alpha}\right), \quad r=1,2,\ldots$$

The probability of failure in a certain time period is:

$$R(t)=e\,(1-e)^{-1}\left[e^{-\left(1+\frac{t}{\beta}\right)^{-\alpha}}-1\right];t>0,\quad \alpha,\beta>0.$$

The survival function $R(t)$ of L-Lomax distribution is decreasing with time t, it is directly related to the scale parameter β when the shape parameter α is fixed, and inversely related with the shape parameter α when scale parameter β is fixed, see Figure 3.

Risk rate function is the ratio of the pdf to the survival function $R(t)$, and of the L-Lomax distribution is :

$$h(t)=\frac{\alpha}{\beta}\left(1+\frac{t}{\beta}\right)^{-(\alpha+1)}\left[e^{\left(1+\frac{t}{\beta}\right)^{-\alpha}}-1\right]^{-1},$$

with cumulative risk rate,

$$H(t)=\ln(e-1)-\ln\left(e^{1-\left(1+\frac{t}{\beta}\right)^{-\alpha}}-1\right).$$

Figure 4 shows the risk function when the scale parameter β is fixed, noted that it is an increasing and then decreasing over the time period, whereas when the shape parameter α is fixed, the risk is a decreasing function, see Figure 5. Therefore, L-Lomax distribution is to be a more flexible model than Lomax distribution through the risk function.

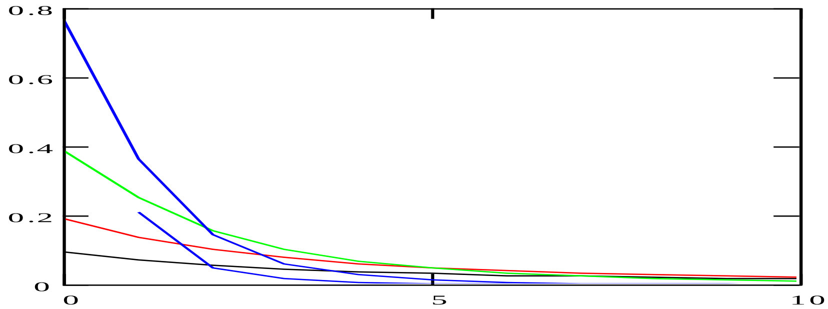

Figure 1 : pdf of L-Lomax distribution.

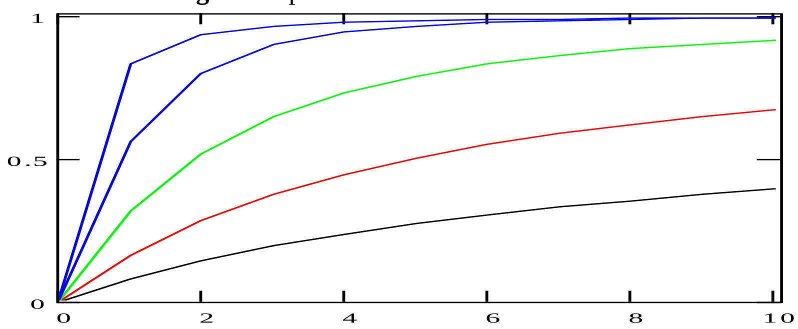

Figure 2 : cdf of L-Lomax distribution.

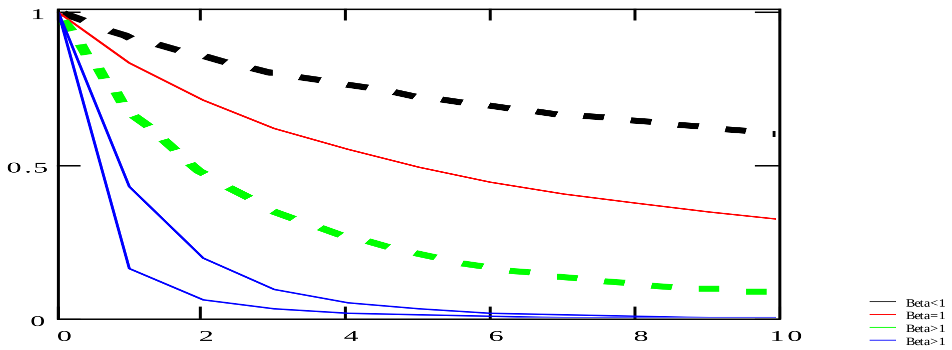

Figure 3 : Survival function of L-Lomax distribution.

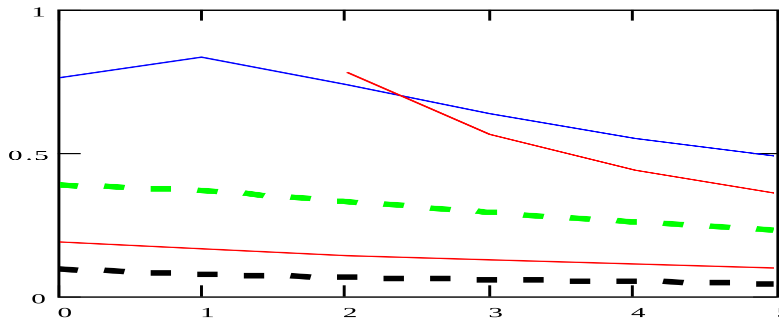

Figure 4 : Risk function of L-Lomax distribution with a fixed scale parameter (β).

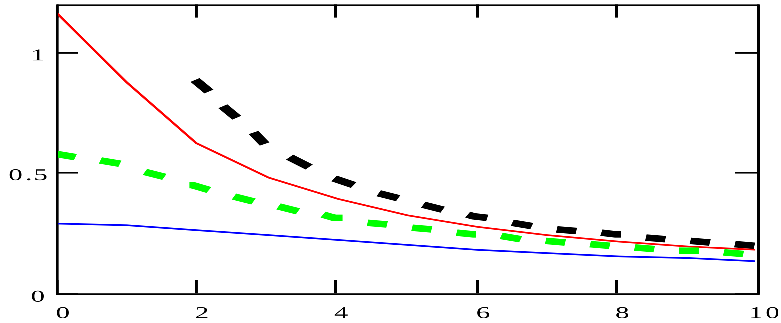

Figure 5 : Risk function of L-Lomax distribution with a fixed shape parameter (α).

Maximum likelihood Estimation : Suppose the a random variable t distributed as the LLO(α,β), then the L.L. function of (α,β) defined as

$$L(\alpha,\beta/t)=\frac{\alpha^n e^n}{(e-1)^n\beta^n}\left\{\prod_{i=1}^{n}\left[\left(1+\frac{t_i}{\beta}\right)^{-(\alpha+1)}\right]\right\}\ e^{-\sum_{i=1}^{n}\left\{\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right\}}, \qquad (4)$$

with log-likelihood function

$$\ln L(\alpha,\beta/t)=n-n\ln(e-1)+n\ln\alpha-n\ln\beta-(\alpha+1)\ \sum_{i=1}^{n}\ln\left(1+\frac{t_i}{\beta}\right)-\sum_{i=1}^{n}\left\{\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right\} \qquad (5)$$

Based on Eq.(5), the non-closed form of $\hat{\alpha}$, $\hat{\beta}$ are getting as

$$\hat{\alpha}=n\left\{\sum_{i=1}^{n}\left[1-\sum_{i=1}^{n}\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right]\ln\left(1+\frac{t_i}{\beta}\right)\right\}^{-1},$$

$$\hat{\beta}=\frac{1}{n}\left\{\sum_{i=1}^{n}\left(1+\frac{t_i}{\beta}\right)^{-1} t_i\left\{\alpha\left[1-\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right]+1\right\}\right\}$$

Therefore, last two equations of $(\hat{\alpha},\hat{\beta})$ will be solved numerically to obtain $(\hat{\alpha},\hat{\beta})$. Following, the fisher information matrix is obtained as

$$\hat{I}=-\begin{bmatrix}\frac{\partial^2\ln L(\alpha,\beta/t)}{\partial\alpha^2} & \frac{\partial^2\ln L(\alpha,\beta/t)}{\partial\alpha\partial\beta}\\ \frac{\partial^2\ln L(\alpha,\beta/t)}{\partial\alpha\partial\beta} & \frac{\partial^2\ln L(\alpha,\beta/t)}{\partial\beta^2}\end{bmatrix}\downarrow(\hat{\alpha},\hat{\beta}),$$

where

$$\frac{\partial^2\ln L(\alpha,\beta/t)}{\partial\alpha^2}=\frac{-n}{\alpha^2}-\sum_{i=1}^{n}\left[\ln\left(1+\frac{t_i}{\beta}\right)\right]^2\left(1+\frac{t_i}{\beta}\right)^{-\alpha}$$

$$\frac{\partial^2 \ln L(t;\alpha,\beta)}{\partial\beta^2}=\frac{n}{\beta^2}-\sum_{i=1}^{n}\left(1+\frac{t_i}{\beta}\right)^{-2}\left(\frac{t_i}{\beta^4}\right)\left\{\alpha[(\alpha+1)t_i+2\beta]\left(1+\frac{t_i}{\beta}\right)^{-\alpha}-(\alpha+1)(2\beta+t_i)\right\}$$

$$\frac{\partial^2 \ln L(t;\alpha,\beta)}{\partial\alpha\partial\beta}=\sum_{i=1}^{n}\left(1+\frac{t_i}{\beta}\right)^{-1}\left(\frac{t_i}{\beta^2}\right)\left[1-\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\left[1+\alpha\ln\left(1+\frac{t_i}{\beta}\right)\right]\right]$$

If the experiment terminated at a specified number of failed units, say $(r,\ r<n)$, then the likelihood function is given by

$$L(t;\alpha,\beta)=\frac{n\,\alpha^r e^n}{(n-r)(e-1)^n\beta^r}\left\{\prod_{i=1}^{r}\left[\left(1+\frac{t_i}{\beta}\right)^{-(\alpha+1)}\right]\right\}\ e^{-\sum_{i=1}^{r}\left\{\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right\}}.\left[1-e^{-\left(1+\frac{t_r}{\beta}\right)^{-\alpha}}\right]^{n-r}, \tag{6}$$

with log-likelihood function

$$\ln L(\alpha,\beta/t)=\ln\left[\frac{n\,e^n}{(n-r)(e-1)^n}\right]+r\ln\alpha-r\ln\beta-(\alpha+1)\sum_{i=1}^{r}\ln\left(1+\frac{t_i}{\beta}\right)-\sum_{i=1}^{r}\left(1+\frac{t_i}{\beta}\right)^{-\alpha}$$
$$+(n-r)\ln\left[1-e^{-\left(1+\frac{t_r}{\beta}\right)^{-\alpha}}\right] \tag{7}$$

Based on Eq. 7 the ML estimates of (α,β) is:

$$\hat{\alpha}=r\left\{\sum_{i=1}^{r}\left(1+T_i^{-\hat{\alpha}}\right)\ln T_i-(n-r)T_r^{-\hat{\alpha}}e^{-T_r^{-\hat{\alpha}}}\left[1-e^{-T_{ri}^{-\alpha}}\right]^{-1}\ln T_r\right\}^{-1}$$

$$\hat{\beta}=\frac{1}{r}\left\{\sum_{i=1}^{r}T_i^{-1}\ t_i\{1+\alpha(1-T_i^{-\alpha})\}+\alpha(n-r)\ t_r T_r^{-(\alpha+1)}e^{-T_r^{-\hat{\alpha}}}\left[1-e^{-T_r^{-\alpha}}\right]^{-1}\right\},$$

where,

$$T_i=\left(1+\frac{t_i}{\beta}\right),\ T_r=\left(1+\frac{t_r}{\beta}\right).$$

To obtain $(\hat{\alpha},\hat{\beta})$, the last two equations of $(\hat{\alpha},\hat{\beta})$ will be solved numerically, and the elements of the Fisher information matrix are

$$\frac{\partial^2 \ln L(\alpha,\beta/t)}{\partial\alpha^2}=\frac{-r}{\alpha^2}-\sum_{i=1}^{r}(\ln T_i)^2T_i^{-\alpha}+(n-r)T_r^{-\hat{\alpha}}e^{-T_r^{-\hat{\alpha}}}\left[1-e^{-T_{ri}^{-\alpha}}\right]^{-1}(\ln T_r)^2T_r^{-\alpha}\left[1-T_r^{-\alpha}\left(1+\left(e^{T_{ri}^{-\alpha}}-1\right)^{-1}\right)\right]$$

$$\frac{\partial^2 \ln L(\alpha,\beta/t)}{\partial\beta^2}=\frac{r}{\beta^2}+\sum_{i=1}^{n}\left(\frac{t_i}{\beta^4T_i}\right)\left\{\left(\alpha+1-\frac{\alpha}{T_i^{\alpha}}\right)\left(\frac{t_i}{T_i}-2\beta\right)-\frac{\alpha^2}{T_i^{\alpha+1}}\right\}$$

$$+\frac{\alpha t_r(n-r)\left(\frac{\alpha t_r}{T_r^{\alpha+1}}\left(\frac{e^{-T_{ri}^{-\alpha}}}{\left(1-e^{-T_{ri}^{-\alpha}}\right)}-1\right)+\frac{(\alpha+1)t_r}{T_r}-2\beta\right)}{\beta^4T_r^{\alpha+1}\left(1-e^{-T_{ri}^{-\alpha}}\right)e^{T_{ri}^{-\alpha}}}$$

$$\frac{\partial^2 \ln L(\alpha,\beta/t)}{\partial\alpha\partial\beta}=\frac{1}{\beta^2}\sum_{i=1}^{n}\frac{t_i}{T_i}\left(1+\frac{(\alpha\ln(T_i)+1)}{T_i^{\alpha}}\right)+\frac{t_r(n-r)}{\beta^2T_r^{\alpha+1}\left(e^{T_{ri}^{-\alpha}}-1\right)}\left[1-\frac{\alpha(\ln T_r)}{T_r^{\alpha}\left(1-e^{-T_{ri}^{-\alpha}}\right)}+\alpha(\ln T_r)\right].$$

Bayesian Estimation In case of complete data (unknown α): Under Eq. 4 β is known and assume the prior of α is gamma (a,b), i.e., $\pi(\alpha)\propto\alpha^{a-1}e^{-b\alpha}$ with α,a,b>0, then the posterior density function of α is:

$$\pi(\alpha/t)\propto\frac{(\alpha)^{n+a-1}}{\left(\beta(e-1)\right)^{n}}\left[\prod_{i=1}^{n}\left(1+\frac{t_i}{\beta}\right)^{-(\alpha+1)}\right]e^{n-b\alpha-\sum_{i=1}^{n}\left\{\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right\}},$$

and for the exponential prior, i.e., $\pi(\alpha)\propto be^{-b\alpha}$ with α,b>0, the posterior density function of α is:

$$\pi(\alpha/t)\propto\frac{b\,(\alpha)^{n}}{(e-1)^{n}(\beta)^{n}}\left\{\prod_{i=1}^{n}\left(1+\frac{t_i}{\beta}\right)^{-(\alpha+1)}\right\}e^{n-b\alpha-\sum_{i=1}^{n}\left\{\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right\}}.$$

Bayesian Estimation In case of incomplete data (known α) : Under Eq. 6 β is known and gamma prior of α, the posterior density function is:

$$\pi(\alpha/t)\propto\frac{1}{(n-r)(e-1)^{n}(\beta)^{r}}\,n\,(\alpha)^{r+a-1}\left(1-e^{-\left(1+\frac{t_r}{\beta}\right)^{-\alpha}}\right)^{n-r}\prod_{i=1}^{r}\left(1+\frac{t_i}{\beta}\right)^{-(\alpha+1)}e^{n-b\alpha-\sum_{i=1}^{r}\left\{\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right\}},$$

and for exponential prior, the posterior density function is:

$$\pi(\alpha/t)\propto\frac{bn\,(\alpha)^{r}}{(n-r)(e-1)^{n}(\beta)^{r}}\left(1-e^{-\left(1+\frac{t_r}{\beta}\right)^{-\alpha}}\right)^{n-r}\prod_{i=1}^{r}\left[\left(1+\frac{t_i}{\beta}\right)^{-(\alpha+1)}\right]e^{n-b\alpha-\sum_{i=1}^{r}\left\{\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right\}}.$$

Therefore, the Bayesian estimator of the parameter α can be obtained as follows

$$\tilde{\alpha}\propto E(\alpha/t)=\int_{\alpha}\alpha\,\pi(\alpha/t)\,d\alpha,$$

numerical simulation to evaluate the value of $\tilde{\alpha}$ is used.

Bayesian Estimation In case of complete data (unknown β): Under Eq. 4 α is known and assume the prior of β is gamma (a,b) distribution, i.e., $\pi(\beta)\propto\beta^{a-1}e^{-b\beta}$ with β,a,b>0, then the posterior density function of β is:

$$\pi(\beta/t)\propto\frac{\alpha^{n}e^{n-b\beta}}{(e-1)^{n}\beta^{n-a+1}}\left\{\prod_{i=1}^{n}\left[\left(1+\frac{t_i}{\beta}\right)^{-(\alpha+1)}\right]\right\}e^{-\sum_{i=1}^{n}\left\{\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right\}},$$

and for the exponential prior, i.e., $\pi(\beta)\propto be^{-b\beta}$ with β,a,b>0, then the posterior density function of β is:

$$\pi(\beta/t)\propto\frac{b\alpha^{n}e^{n-b\beta}}{(e-1)^{n}\beta^{n}}\left\{\prod_{i=1}^{n}\left[\left(1+\frac{t_i}{\beta}\right)^{-(\alpha+1)}\right]\right\}e^{-\sum_{i=1}^{n}\left\{\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right\}}.$$

Bayesian Estimation In case of incomplete data (known β): Under Eq. 6 α is known and gamma prior of β, the posterior density function of β is:

$$\pi(\beta/t)\propto\frac{n\,(\alpha)^{r}\left(1-e^{-\left(1+\frac{t_r}{\beta}\right)^{-\alpha}}\right)\left(\prod_{i=1}^{r}\left(1+\frac{t_i}{\beta}\right)^{-(\alpha+1)}\right)e^{n-b\beta-\sum_{i=1}^{r}\left\{\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right\}}}{(n-r)(e-1)^{n}(\beta)^{r-a+1}},$$

and for the exponential prior, i.e., $\pi(\beta)\propto be^{-b\beta}$ with β,a,b>0, then the posterior density function of β is:

$$\pi(\beta/t)\propto\frac{bn\,(\alpha)^{r}\left(1-e^{-\left(1+\frac{t_r}{\beta}\right)^{-\alpha}}\right)\left(\prod_{i=1}^{r}\left(1+\frac{t_i}{\beta}\right)^{-(\alpha+1)}\right)e^{n-b\beta-\sum_{i=1}^{r}\left\{\left(1+\frac{t_i}{\beta}\right)^{-\alpha}\right\}}}{(n-r)(e-1)^{n}(\beta)^{r}}.$$

Therefore, the Bayesian estimator of the parameter β can be obtained as follows

$$\tilde{\beta} \propto E(\beta/t) = \int_{\beta} \beta\, \pi(\beta/t)\, d\beta,$$

numerical simulation to evaluate the value of $\tilde{\beta}$ is used.

Results and discussion

Life data from the L-Lomax distribution is generated using Math-Cad software for getting L-Lomax parameter estimations with some properties by applying non-Bayesian and Bayesian approach based on complete and incomplete data. Starting with the two Markov chains $(\alpha_0 = \beta_0 = 0.5)$, the summary of the sampling results concerning the unknown parameters (α, β) for non-Bayesian complete and incomplete data are displayed in Tables 1 to 4. The Monte Carlo simulations for different random samples ranging from 500 to 40000 for each Markov chain are running, and the accuracy of the estimators is calculated in terms of Monte Carlo standard error (MC error) of the mean according to (Spiegelhalter et al., 2003).

Table 1 Non-Bayesian estimation of scale parameter β in case of complete data.

Init. Val.	mean	sd	MC error	2.5%	median	97.5%	sample
	0.5060	0.2827	0.014750	0.01737	0.5072	0.9743	500
	0.5076	0.2899	0.009305	0.01905	0.5072	0.9783	1000
	0.5011	0.2897	0.006441	0.02169	0.5006	0.9772	2000
$\alpha_0 = 0.5$	0.5023	0.2907	0.002753	0.0215	0.5032	0.9755	10000
	0.5006	0.2902	0.001995	0.02324	0.5006	0.9757	20000
	0.4999	0.2892	0.001501	0.02451	0.4989	0.9747	40000

Table 2 Non-Bayesian estimation of shape parameter α in case of complete data.

Init. Val.	mean	sd	MC error	2.5%	median	97.5%	sample
$\beta_0 = 0.5$	0.4981	0.2797	0.012390	0.04474	0.5037	0.97	500
	0.4960	0.2901	0.009272	0.01878	0.4975	0.9738	1000
	0.5012	0.2886	0.007272	0.02461	0.506	0.9751	2000
	0.5023	0.2898	0.003021	0.02361	0.5045	0.9764	5000
	0.5002	0.2891	0.001975	0.02386	0.4987	0.9747	10000
	0.5014	0.2898	0.001340	0.02471	0.5001	0.9757	20000

Table 3 Non-Bayesian estimation of scale parameter β in case of incomplete data $(r = 3)$.

Init. Val.	mean	sd	MC error	2.5%	median	97.5%	sample
$\alpha_0 = 0.5$	0.5062	0.2801	0.010810	0.01912	0.5061	0.9743	500
	0.5076	0.2899	0.009305	0.01905	0.5072	0.9783	1000
	0.5011	0.2897	0.006441	0.02169	0.5006	0.9772	2000
	0.5006	0.2907	0.003883	0.02114	0.5036	0.9783	5000
	0.5023	0.2907	0.002753	0.0215	0.5032	0.9755	10000
	.50060	0.2902	0.001995	0.02324	0.5006	0.9757	20000

Table 4 Non-Bayesian estimation of shape parameter α in case of incomplete data $(r = 3)$.

Init. Val.	mean	sd	MC error	2.5%	median	97.5%	sample
$\beta_0 = 0.5$	0.5026	0.2867	0.013360	0.02515	0.506	0.9715	500
	0.4960	0.2901	0.009272	0.01878	0.4975	0.9738	1000
	0.5012	0.2886	0.007272	0.02461	0.506	0.9751	2000
	0.5058	0.2894	0.004286	0.02395	0.5059	0.976	5000
	0.5023	0.2898	0.003021	0.02361	0.5045	0.9764	10000
	0.5002	0.2891	0.001975	0.02386	0.4987	0.9747	20000

For Bayesian estimation, Assume that the experiment is terminated at a specified number of failure units $(r = 5)$ where $(n = 15)$. starting with the Markov chain with initial values $(\alpha_0 = 0.5,\ \beta_0 = 0.25)$ in case of complete data and $(\alpha_0 = 0.75, \beta_0 = 0.75)$ in case of incomplete data.

case I : assume the value of the parameter β is known with various priors of the parameter α.

case II : assume the value of the parameter α is known with various priors of the parameter β.

The iterations are run 50000 for each Markov chain, and the posterior mean, median and standard deviation with a 95% posterior credible interval are displayed in Tables 5 to 8.

Table 5 Bayesian estimation of shape parameter α in case of complete data (**case I**).

Prior	mean	sd	MC error	2.5%	median	97.5%	iteration
Weibull(2,2)	0.6263	0.329	0.001441	0.1112	0.5893	1.367	50000
Beta(0.5,0.5)	0.4986	0.3543	0.001574	0.001499	0.495	0.9985	50000
Exp(2)	0.5005	0.5026	0.002252	0.01236	0.3473	1.867	50000
Chisqr(0.5)	0.495	0.9878	0.004455	5.033E-7	0.0867	3.423	50000
Gamma(0.5,0.5)	0.9919	1.401	0.006158	9.197E-4	0.4473	4.998	50000

Table 6 Bayesian estimation of scale parameter β in case of complete data (**case II**).

Prior	mean	sd	MC error	2.5%	median	97.5%	iteration
Beta(2,3)	0.4001	0.2	8.684E-4	0.06875	0.3864	0.806	50000
Exp(3)	0.3336	0.335	0.001501	0.008237	0.2315	1.245	50000
Weibull(1.5,1.5)	0.6887	0.4698	0.002068	0.06474	0.5985	1.837	50000
Gamma(2,3)	0.6667	0.4688	0.002133	0.08324	0.5615	1.849	50000
Chisqr(2.5)	2.512	2.255	0.01037	0.1192	1.876	8.461	50000

Table 7. Bayesian estimation of shape parameter α in case of incomplete data with $(n = 15, r = 5)$ (**case I**).

Prior	mean	sd	MC error	2.5%	median	97.5%	iteration
Weibull(2,2)	0.6263	0.329	0.001441	0.1112	0.5893	1.367	50000
Exp(3)	0.3336	0.335	0.001501	0.008237	0.2315	1.245	50000
Beta(0.5,0.5)	0.4986	0.3543	0.001574	0.001499	0.495	0.9985	50000
Chisqr(0.5)	0.495	0.9878	0.004455	5.033E-7	0.0867	3.423	50000
Gamma(0.5,0.5)	0.9919	1.401	0.006158	9.197E-4	0.4473	4.998	50000

Table 8. Bayesian estimation of scale parameter β in case of incomplete data with $(n = 15, r = 5)$ (**case II**).

Prior	mean	sd	MC error	2.5%	median	97.5%	iteration
Beta(4,4)	0.5001	0.1661	7.196E-4	0.1863	0.4998	0.8148	50000
Weibull(3,4)	0.5621	0.2054	8.982E-4	0.1835	0.5579	0.9774	50000
Exp(4)	0.2502	0.2513	0.001126	0.006178	0.1737	0.9337	50000
Gamma(4,4)	0.9975	0.499	0.002342	0.2769	0.9154	2.191	50000
Chisqr(1)	0.9919	1.401	0.006158	9.197E-4	0.4473	4.998	50000

From Tables 3 and 4, it is observed the MC error for each node is less than 5% of the sample standard deviation. In case of complete and incomplete data, Bayes estimator corresponding to Weibull prior represent the best Bayes estimator of the parameter αas well as Bayes estimator corresponding to Beta prior represent the best Bayes estimator of the parameter βcomparing to other Bayes estimators. The shape of the posterior densities of α and βcan be shown from Figures 6 and 7.

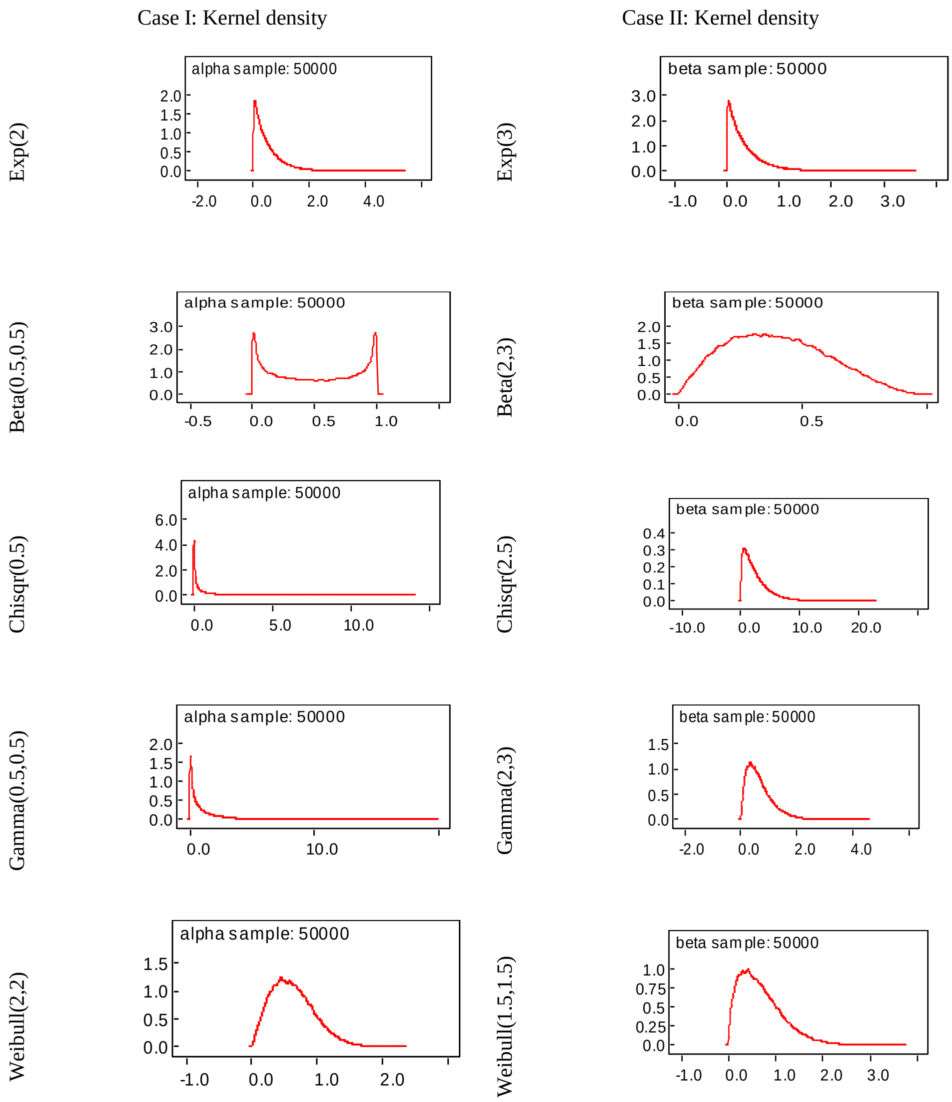

Figure 6: Kernel density with various prior densities in case of complete data.

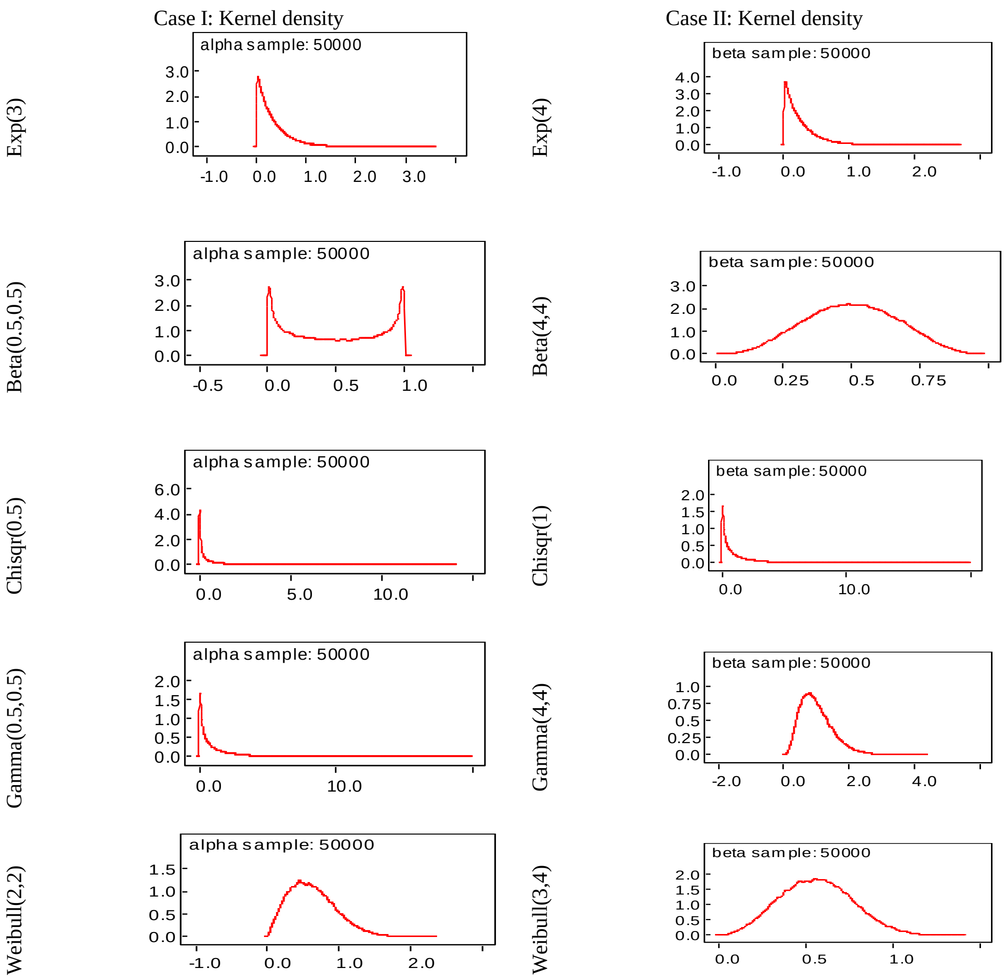

Figure 7: Kernel density with various prior densities in case of incomplete data.

Conclusion

L-Lomax distribution is introduced in this paper using DUS method proposed by [3]. Row moments, probability of failure, risk rate function and cumulative risk rate are provided. The survival function of the L-Lomax is decreasing with time t, and it is directly related to the scale parameter β when the shape parameter α is fixed, and inversely related with the shape parameter α when the scale parameter β is fixed. Also, it is noticed, the risk function when the scale parameter β is fixed is an increasing and then decreasing over the time period, whereas when the shape parameter α is fixed, it is a decreasing function. Therefore, L-Lomax distribution is to be a more flexible model than Lomax distribution through the risk rate function. Maximum likelihood and Bayesian estimation based on complete and incomplete data are provided. Monte Carlo simulations with L-Lomax generated data ranging from 500 to 50000 are running. The results show the MC error for each parameter is less than 5%, and its decreasing as the sample size increases. As well as, various informative priors (exponential, Weibull, gamma, beta and chi-square) are used, Bayes estimator corresponding to Weibull prior represents the best Bayes estimator of the parameter α and Bayes estimator corresponding to Beta prior represents the best Bayes estimator of the parameter β comparing to other Bayes estimators.

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