African Journal of Advanced Pure and Applied Sciences (AJAPAS)
Online ISSN: 2957-644X
Volume 4, Issue 4, 2025
Page No: 334-338
Website: https://aaasjournals.com/index.php/ajapas/index

ISI 2025: 1.126 SJIFactor 2024: 6.752 معامل التأثير العربي: 1.62

The Pareto-Bernoulli Equation: A New Theoretical Framework for Multi-Objective Optimization Based on Energy Conservation Principles

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معادلة باريتو-برنولي: إطار نظري جديد لتحسين متعدد الأهداف استنادًا إلى مبادئ الحفاظ على الطاقة

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Received: August 16, 2025	Accepted: October 21, 2025	Published: October 30, 2025

Abstract:
This paper introduces a new mathematical concept that bridges two seemingly heterogeneous domains: multi-objective nonlinear programming (MONLP) and Bernoulli's equation in fluid dynamics. Inspired by the fundamental structure of Bernoulli's equation—which describes energy conservation in steady flow—we propose a new model for analyzing Pareto fronts in MONLP problems with non-convex, dynamically coupled functions. We propose a mathematical analogy where the conflicting objectives in MONLP represent the different energy terms in the Bernoulli equation (kinetic energy, pressure energy, and potential energy), and the global "Bernoulli constant" is represented as a Pareto-Sum Constant that indicates the global optimal equilibrium state of the multi-objective system. We present a new mathematical formulation called the Pareto-Bernoulli Equation (PBE) and an associated transformation technique. This research opens new avenues for physics-inspired optimization algorithms and provides a unified theoretical perspective on Pareto optimality through energy conservation principles.

Keywords: Multi-objective nonlinear programming, Bernoulli's equation, Pareto front.

الملخص
تُقدّم هذه الورقة مفهومًا رياضيًا جديدًا يربط بين مجالين يبدوان غير متجانسين: البرمجة غير الخطية متعددة الأهداف (MONLP) ومعادلة برنولي في ديناميكا الموائع. واستلهامًا من البنية الأساسية لمعادلة برنولي التي تصف حفظ الطاقة في التدفق الثابت - نقترح نموذجًا جديدًا لتحليل جبهات باريتو في مسائل البرمجة غير الخطية متعددة الأهداف (MONLP) ذات الدوال غير المحدبة والمقترنة ديناميكيًا. نقترح تشبيهًا رياضيًا حيث تُمثّل الأهداف المتضاربة في MONLP حدود الطاقة المختلفة في معادلة برنولي (الطاقة الحركية، وطاقة الضغط، والطاقة الكامنة)، ويُمثّل "ثابت برنولي" العالمي بثابت مجموع باريتو الذي يُشير إلى حالة التوازن الأمثل العالمي للنظام متعدد الأهداف. نقدم صياغة رياضية جديدة تسمى معادلة باريتو-برنولي (PBE) وتقنية التحويل المرتبطة بها. يفتح هذا البحث آفاقًا جديدة لخوارزميات التحسين المستوحاة من الفيزياء ويوفر منظورًا نظريًا موحدًا حول تحسين باريتو من خلال مبادئ الحفاظ على الطاقة.

الكلمات المفتاحية: البرمجة الغير خطية متعددة الأهداف، معادلة برنولي، باريتو فرونت.

1. Introduction

Multi-objective nonlinear programming (MONLP) is a fundamental component of current decision-making, solving problems with competing objectives in engineering, economics, and operations research. The main challenge is to identify Pareto-optimal solutions that represent the optimal balance between conflicting objectives [1]. Traditional approaches, such as benchmarking methods and evolutionary algorithms [2,3,4], often struggle with non-convex Pareto fronts and complex objective interactions.

At the same time, Bernoulli's equation in fluid dynamics shows how energy is conserved by showing how multiple kinds of energy (kinetic, potential, and pressure) keep the total energy along streamlines the same [5]. This physical concept has been utilized in numerous engineering fields, but has yet to be investigated in mathematical optimization.

Although considerable research has been conducted in multi-objective optimization [1,2,6] and energy-based modeling [5,7,11], there is currently no work that integrates these fields using a rigorous mathematical framework. Existing multi-objective approaches [3,8] do not provide physical explanations for Pareto front configurations, and energy conservation laws continue to be restricted to modeling physical systems.

This paper makes three primary contributions:

1. **Theoretical Innovation**: We introduce the Pareto-Bernoulli Equation (PBE) as a novel mathematical framework linking energy conservation to Pareto optimality.

2. **Rigorous Foundation**: We establish formal conditions for energy-based transformations Φ_i and prove the existence and uniqueness of the Pareto-Sum constant C_{ps}.

3. **Practical Implications**: We demonstrate how PBE provides new insights into non-convex Pareto front structures and enables innovative algorithm design.

Our work builds upon fundamental concepts from multi-objective optimization [1,2,9] and mathematical transformations [6,10], extending them through energy conservation principles to create a unified theoretical perspective.

2. MONLP and Bernoulli's Equation

MONLP deals with the simultaneous optimization of multiple conflicting objective functions under constraints and is fundamental in decision science, engineering, and economics. The main challenge is to find the Pareto optimal solution set and its Pareto front. On the other hand, the Bernoulli equation, a direct consequence of the energy conservation principle in fluid mechanics, describes the relationship between velocity (v), pressure (p), and height (z) in an ideal, steady-state, incompressible flow:

$$\boldsymbol{(1/2)\rho v^2 + \rho g z + p = constant} \quad \textbf{(Bernoulli's Equation)}$$

where $\boldsymbol{\rho}$ is the density, and g is the acceleration due to gravity. The constant represents the total mechanical energy per unit volume. This paper introduces a new idea: interpreting the Bernoulli constant as a Pareto-Sum Constant in the objective space for MONLP problems with a dynamical or energetic potential structure.

2.1 New mathematical idea: Pareto-Bernoulli equation (PBE):

Consider a MONLP problem with k objectives:

$$\boldsymbol{minF(x) = \{f_1(x), f_2(x), f_3(x) \dots . f_k(x)\}, x \in S}$$

;S the decision space.

The basic idea is to assume a mathematical transformation ϕ (similar to Legendre or energy transformations) that connects the decision space S to a new objective space Ω, where:

Bernoulli-like terms: Each objective function $f_i(x)$ is reformulated as an "energy" term in $\Omega-space: \phi_i(f_i(x)).\ \phi$ must retain the properties of conservation and exchange (just as velocity, pressure, and altitude exchange energy in Bernoulli).

Pareto constant sum (PSC): There is a constant C_{ps} (which depends on the problem and Φ) such that the Pareto-optimal solutions x^* have the property:

$\sum_{i=1}^{k} \boldsymbol{\phi}_i\,(\boldsymbol{f}_i(\boldsymbol{x}^*)) = \boldsymbol{C}_{ps}$ **Pareto-Bernoulli Equation (PBE)**

Analogy: Just as the sum of Bernoulli energies remains constant along a streamline, the transformed sum of Pareto-optimal objectives remains constant C_{ps} along a Pareto-optimal frontier in Ω -space. C_ps represents the constant "total cost/energy" ratio of a multicriteria system in its optimal equilibrium.
Non-convex dynamics: The strength of this proposal lies in its handling of non-convexity. Just as the Bernoulli equation describes non-linear behavior (increasing velocity leads to a decrease in pressure), PBE represents the complex exchanges (curves, gaps) in the Pareto front of non-convex MONLP problems as an expression for the conservation of C_{ps} under a ϕ transformation.

3. Theoretical basis and the $\boldsymbol{\phi}$ transformation:
We propose that ϕ is a transformation that reflects the "compression" or "tension" of the target space to extract the conservation structure. One of the proposed formulas is inspired by Bernoulli's formula:

$$\boldsymbol{\phi}_i(\boldsymbol{f}_i(\boldsymbol{x})) = \boldsymbol{\alpha}_i\,(\mathbf{f}_i(\mathbf{x}))^{\boldsymbol{\beta}_i}$$

where α_i, β_i are transformation coefficients determined based on the nature of target i and its relationship to other systems. $\boldsymbol{\phi}$ must satisfy:

a) Monotony: $\phi_i$monotonically increasing/decreasing with f_i.
b) Normalized Trade-off: The derivative of $\boldsymbol{\phi}$ should allow a clear definition of the marginal exchange rate between the transferred targets.
c) Conservation: The sum $\phi_i(f_i(x))$is constant only for Pareto-optimal solutions x*.

3.1Theorem (PBE core): For multi-objective problems with an underlying energy structure under the appropriate. $\boldsymbol{\phi}$ transformation, then:

a) Pareto-optimal solutions $\boldsymbol{x}^*$ satisfy the Pareto-Bernoulli equation:
$$\sum_{i=1}^{k} \boldsymbol{\phi}_i\,(\boldsymbol{f}_i(\boldsymbol{x}^*)) = \boldsymbol{C}_{ps}$$
b) The constant $\boldsymbol{C}_{ps}$ is unique for each Pareto optimal front.
c) any solution that achieves PBE is a Pareto optimal solution.

3.2 Mathematical proof
Hypotheses

- the objective functions are continuously differentiable,
- the transformation$\boldsymbol{\phi}_i\colon \boldsymbol{R} \longrightarrow \boldsymbol{R}$ is monotonic (for the functions to be minimized) and differentiable,
- there is a power function
  $\boldsymbol{H}(\boldsymbol{x}) = \sum_{i=1}^{k} \boldsymbol{\phi}_i\,(\boldsymbol{f}_i(\boldsymbol{x}))$ representing the system
- the Pareto front is continuous and consists of non-dominated points.

Mathematical Tools:

- Lagrange's Principle for Optimal Solutions:
$$\boldsymbol{\nabla f}_i(\boldsymbol{x}^*) + \sum_{i \neq j}^{k} \boldsymbol{\lambda}_j\,\boldsymbol{\nabla f}_j(\boldsymbol{x}^*) = \boldsymbol{0}, \boldsymbol{\lambda}_j \geq \boldsymbol{0}.$$
- Implicit derivative theorem for analyzing the structure of the Pareto front.

Part 1: Every Pareto-optimal solution satisfies PBE
Let's assume that $\boldsymbol{x}^*$ is a Pareto-optimal solution. By definition, there is no x that improves one objective without harming another.

We define the transformed energy function:
$\boldsymbol{H}(\boldsymbol{x}) = \sum_{i=1}^{k} \boldsymbol{\phi}_i\,(\boldsymbol{f}_i(\boldsymbol{x}))$

In x∗, the gradient of H is a coefficient of the gradients of the objectives.

$$\nabla \boldsymbol{H}(\boldsymbol{x}^*) = \sum_{i=1}^{k} \frac{\partial \boldsymbol{\Phi}_i}{\partial \boldsymbol{f}_i} \nabla \boldsymbol{f}_i(\boldsymbol{x}^*)$$

From Lagrange's principle for multi-objective optimization solutions

$$\exists\, \boldsymbol{\lambda}_i \geq \mathbf{0}\ \boldsymbol{s.t} \sum_{i=1}^{k} \boldsymbol{\lambda}_i \nabla \boldsymbol{f}_i(\boldsymbol{x}^*) = \mathbf{0}$$

If we choose $\boldsymbol{\phi}_i$ such that

$$\frac{\partial \boldsymbol{\Phi}_i}{\partial \boldsymbol{f}_i} = \boldsymbol{\lambda}_i\, , \forall_i$$

becomes:

$$\nabla \boldsymbol{H}(\boldsymbol{x}^*) = \sum_{i=1}^{k} \boldsymbol{\lambda}_i \nabla \boldsymbol{f}_i(\boldsymbol{x}^*) = \mathbf{0}.$$

This means that H(x) is constant in the vicinity of$\boldsymbol{x}^*$ (using the implicit function theorem).

So, Because of the Pareto front's connectivity, the constant extends to the entire front:

$$\sum_{i=1}^{k} \boldsymbol{\phi}_i\,(\boldsymbol{f}_i(\boldsymbol{x}^*)) = \boldsymbol{C}_{\boldsymbol{ps}}\, , \forall \boldsymbol{x}^* \in \ \mathbf{PF}$$

Part 2: The constant $\boldsymbol{C}_{\boldsymbol{ps}}$ is unique for each Pareto front.

Assume that there are two constants $\boldsymbol{C}_1, \boldsymbol{C}_2$ for the same Pareto front.

Because of the PF connection, there is a connected path between any two points on the front.

$\boldsymbol{H}(\boldsymbol{x}) = \sum_{i=1}^{k} \boldsymbol{\phi}_i\,(\boldsymbol{f}_i(\boldsymbol{x}))$ is continuous and differentiable on this path.
but
∇H=0 on the front (from Part 1), so H is constant over the entire path
$\boldsymbol{C}_1 = \boldsymbol{C}_2 = \boldsymbol{constant}$
So
$\boldsymbol{C}_{\boldsymbol{ps}}$ is unique for each Pareto front

Part 3: Any solution that satisfies PBE is Pareto optimal
Assume that $\breve{x}$ satisfies$\sum_{i=1}^{k} \boldsymbol{\phi}_i\,(\boldsymbol{f}_i(\breve{x})) = \boldsymbol{C}_{\boldsymbol{ps}}$ but it is not Pareto optimal.
So there is an $\ddot{x}$ Dominates $\breve{x}$:

$$\boldsymbol{f}_i(\ddot{x}) \leq \boldsymbol{f}_i(\breve{x}) \forall_i, \boldsymbol{f}_j(\ddot{x}) < \boldsymbol{f}_j(\breve{x})\ \boldsymbol{for\ some\ j}.$$

$$\boldsymbol{\phi}_i(\boldsymbol{f}_i(\ddot{x})) \geq \boldsymbol{\phi}_i(\boldsymbol{f}_i(\breve{x}))\ \forall_i, \qquad \boldsymbol{\phi}_j(\boldsymbol{f}_j(\ddot{x})) > \boldsymbol{\phi}_j(\boldsymbol{f}_j(\breve{x}))\ \boldsymbol{for\ some\ j}$$

So
$\boldsymbol{H}(\ddot{x}) = \sum_{i=1}^{k} \boldsymbol{\phi}_i\,(\boldsymbol{f}_i(\ddot{x})) > \sum_{i=1}^{k} \boldsymbol{\phi}_i\,(\boldsymbol{f}_i(\breve{x})) = \boldsymbol{C}_{\boldsymbol{ps}}$

But on the Pareto optimal front PF , we have $\boldsymbol{H}(\boldsymbol{x}^*) = \boldsymbol{C}_{\boldsymbol{ps}}$ (from Part 1).
this means that $\ddot{x} \notin PF$,

but $\boldsymbol{H}(\ddot{\boldsymbol{x}}) > \boldsymbol{C}_{\boldsymbol{ps}}$ contradicts the fact that $\boldsymbol{C}_{\boldsymbol{ps}}$ represents the minimum energy

The paradox proves that $\breve{x}$ must be Pareto optimal solution

$$\breve{x} \in \boldsymbol{PF}$$

3.3 An illustrative example of proof:

$$f_1(x) = \frac{1}{2}x^2, \;\; f_2(x) = \frac{1}{2}(2 - x^2), x \in [0,2]$$

The Transformation $\boldsymbol{\phi_i}(f_i) = \sqrt{2f_i}$.

The power function : $\boldsymbol{H}(\boldsymbol{x}) = \sqrt{2f_1} + \sqrt{2f_2} = \boldsymbol{x} + (2 - \boldsymbol{x}) = \mathbf{2}$

Pareto solutions**:** $\boldsymbol{x} \in [\mathbf{0}, \mathbf{2}]$

$$\nabla \boldsymbol{H} = \frac{\partial H}{\partial x} = 1 - 1 = 0$$

Any point satisfies $\boldsymbol{H}(\boldsymbol{x}) = \mathbf{2}$ **belongs to PF**

Significance of the Proof

1. Mathematical Tuning: Links fluid mechanics and multi-objective optimization via the conservation principle.
2. Guiding Algorithms: Finding instead of the Pareto front directly.
   Using $\nabla \boldsymbol{H} = 0$ as a stopping condition.
3. Interpretation of Non-Convexity: Pareto front curvature $\propto \|\nabla^2 H\|$.

Conclusion

The Pareto-Bernoulli framework represents a paradigm shift in understanding the mathematical structure of multi-objective optimization. By combining a fundamental physical principle (energy conservation) with one of the most important concepts of operations research (Pareto optimality), this research opens up new avenues for interdisciplinary research. The results presented here not only provide an original mathematical solution but also lay the foundation for a new school of thought in the design of physics-inspired optimization algorithms. The proof opens new horizons for integrating physics into multi-objective optimization, with applications in:

- Engineering: Network resource allocation.
- Economics: Multi-criteria market equilibrium.
- Artificial Intelligence: Training multi-task models

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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