African Journal of Advanced Pure and Applied Sciences (AJAPAS)

Online ISSN: 2957-644X
Volume 4, Issue 4, 2025
Page No: 707-711
Website: https://aaasjournals.com/index.php/ajapas/index

ISI 2025: 1.126 **SJIFactor 2024: 6.752** **معامل التأثير العربي: 1.62**

Some Remarks on p'-Lindelöf Bitopological Spaces

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بعض خصائص ليندلوف في الفضاءات التبولوجية الثنائية

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Received: September 28, 2025	Accepted: December 05, 2025	Published: December 16, 2025

Abstract:
This paper is a presentation to generalize the notion of Lindelöfness in pairwise topological space, we first obtain a new concept of pairwise Lindelöf bitopological space according to $\tau\sigma$-open sets in [7], called p'-**Lindelöf** (BTS), under this approach many results could be obtained by examining the relationships between separation axioms, lindelöfness and compactness on pairwise topological spaces.

Keywords: bitopological spaces, p'-compact, p'-lindelöf.

الملخص
تعد خاصية ليندلوف تعميما للتضاغط في الفضاءات التبولوجية بوجه عام وكذلك بالنسبة للتبولوجيا الثنائية، في هذا البحث ومن خلال دراسة المجموعات المفتوحة التي تم تعريفها في [7] - سيعرض تعميما لخاصية ليندلوف في الفضاءات الثنائية وستسمى ليندلوف الثنائية، ثم ستناقش هذه الخاصية في حالة مسلمات الفصل،الفضاءات المتراصة وكذلك في حالة الدوال المستمرة. بناء علي هذا الاطار التبولوجي ،العديد من النتائج يمكن استخلاصها كربط الفضاءين الثانئيين المنتظم وفيرشيه مع خاصية ليندلوف الثنائية لتكوين الفضاء الثنائي العادي.

الكلمات المفتاحية: التبولوجيا الثنائية، التراص في الفضاءات الثنائية، فضاء ليندلوف الثنائي.

Introduction

In the early 1960s Kelly introduced the structure of bitopological space bitopological space (X, τ, σ) as a set X adapted with two distinct topological spaces τ and σ [1].
Following Kelly's framework, many topologists have extended classical topological concepts such as separation axioms [2], [3], [4], compactness [5] and lindelöfness [6] to the sitting of bitopological spaces.
In this direction, several bitopological analogous of fundamental topological notions have been systematically developed; these include open and closed sets, interior, closure operators, as well as various forms of compactness and Lindelöfness, all adopted to bitopological space.
The principle objective of this paper, is to establish a new formulation of Lindelöfness within the context of bitopological space which defined in [7], and to investigate its relationship between p'- compactness, p'- Lindelöfness, lindelofness, and compactness.
For clarity, it will be referred to our definition of pairwise lindelöf as p'-lindelöf; to distinguish it from the earlier definition, presented in [6].
It is worth noting that, the additional discussion and earlier result, relating to bitopological spaces could be found in [7]. Furthermore, the reader should note that the notion of the related concept in bitopological space such as continuous function, separation axioms, compactness, and lindelöfness in bitopological space will be denoted by (p') instead of (p); due to the base definition of open set in bitopological space, have taken from [7] which stronger than in which other reference.

Paper outline:
This paper is divided into three sections:
Frist section recounts the relevant material on pairwise topological spaces.
In second section, a new definition of lindelöfness in pairwise topological space will be formulated by essential concepts which were pretended in [7], then discuses many topological properties such as compactness and lindelöfness.
The third section presents a summary of the results, obtained in the second section.

Preliminaries

This section is a brief introduction to some background material on the notion of bitopological spaces which may need in the following part of the paper.

According to [3], triple (X, τ, σ) is referred as **pairwise T_1 space** (denote p'-T_1) if, for any non-equal points $a, b \in X$ there exist τσ-open sets O_1, O_2 such that $x \in O_1, y \notin O_1, y \in O_2, x \notin O_2$.

As defined in [7], a bitopological space (X, τ, σ) is said to be a **pairwise regular space (**denoted p'-regular) if for every τσ-closed set A and any point with $b \notin A$, there exist τσ-open sets O_1 and O_2 having a void intersection such that $b \in O_1$ and $A \subseteq O_2$.

Furthermore, [7] introduced the concept of a **pairwise normal space** (denoted p'-normal) as a following: if every pair of its disjoint τσ-closed sets E and F, there exist disjoint τσ-open setsO_1, O_2 such that $E \subseteq O_1$ and$F \subseteq O_2$.

Then, a p'-topological space, is called **p'-T_3 space**, if it satisfies both p'-regular and p'-T_1 condition. Likewise, it is called a **p'-T_4 space** if it holds both p'-normal and p'-T_1 space properties.

In [3], a mapping $f: (X, \tau_1, \sigma_1) \longrightarrow (Y, \tau_2, \sigma_2)$ between two bitopological spaces is defined to be a **pairwise continuous** (denoted p'- continuous) if, for $V \in \tau_2 \cup \sigma_2$, the preimage $f^{-1}(V)$ belongs to $\tau_1 \cup \sigma_1$.

According to [7], whenever both (X, τ) and (X, σ) are compact spaces, then a bitopological space (X, τ, σ) is named **compact**.

In classical topology, it is straightforward to verify that, any compact space is lindelöf; since each open cover can be reduced to a finite subcover, and finite sets a countable. More details about general topology can be seen in [8].

In addition, a collection ℧ of $\tau\sigma$-open O of X, is defined as a **pairwise open cover** (denoted p'-open cover) whenever $X \subseteq \cup \mathrm{O}_{O \in ℧}$ [7].

Finally, if any p'-open cover of X admits a finite subcover in a bitopological space (X, τ, σ)then it is termed **pairwisecompact** (denoted, p'-compact) [5].

Results and discussion

lindelöfness in bitopological spaces

This section focuses on the development of the lindelöfness property within the setting of bitopological spaces, and explores its relationship with compactness in both classical topology and bitopology.

In addition, we examine whether lindelöfness is preserved under image of continuous maps, and whether it is inherited by subspaces in bitopological spaces.

Definition1. In bitopological space, a triple (X, τ, σ) is termed a lindelöf space, whenever both (X, τ) and (X, σ) are lindelöf.

Definition2. In bitopological space (X, τ, σ) if any p'-open cover of X admits a countable subcover, then It is called a pairwise lindelöf space (denote p'-lindelöf)

Example1. If $(X, \tau), (X, \sigma)$ are a trivial topology and discrete topology respectively on $\mathbb{R}$, then (X, τ, σ) is not p'-lindelöf. Although, where $X = \mathbb{N}$ then (X, τ, σ) holds a p'-lindelöfness propraty.

Corollary1. A p'-compactness property implies p'-lindelöfness in p'- topological spaces.

Proof. Trivial (by definition 2 and the definition of p'-compactness)■

The following example shows, the converse implication of Corollary1 does not always hold.

Example 2. Assume that $X = \mathbb{R}$ with the trivial space (X, τ) and the usual space (X, σ), then (X, τ, σ) holds p'-lindelöfness and does not hold p'-compactness.

Theorem1. A bitopological space is lindelöf, if it is p'-lindelöf.

Proof. Assume to the contrary, that (X, τ, σ) is not a lindelöf space. It follows that, at least one of (X, τ) or (X, σ) is non-lindelöf, suppose (X, τ) fails to be lindelöf, then there exists open cover $\mho$ of X can not yield any countable subcover. Therefore (X, τ, σ) fails to be p'-lindelöf.■

In general, the reverse implication of Theorm1 does not always hold, see the following example:

Example3. If $(\mathbb{R}, \tau_L)$ a Sorgenfrey line space and $(\mathbb{R}, \tau_R)$ a mirror Sorgenfrey line space on $\mathbb{R}$, with lindelöf spaces $(\mathbb{R}, \tau_L)$ and $(\mathbb{R}, \tau_R)$,then $(\mathbb{R}, \tau_L, \tau_R)$ is lindelöf, while it is not p'-lindelof, since $(\mathbb{R}, \tau_L, \tau_R)$ the discrete topology.

Definition3. A subset B of a bitopological space (X, τ, σ) is p'- lindelöf if (B, τ_B, σ_B) is p'-lindelöf as a sub bitopological space.

Theorem2. In p'-lindelöf space, any τσ-closed set is p'-lindelof.

Proof. Assume that $\{O_\beta\}_{\beta\in I}$ is a p'- open cover of a τσ-closed set B, then $B^c \cup \{O_\beta\}_{\beta\in I}$ is a p'-open cover for X, by p'-lindelöfness, $B^c \cup \{O_\beta\}_{\beta\in I}$ admits a countable subcover $\{B^c, O_{\beta 1}, O_{\beta 2}, O_{\beta 3}, \dots\}$, that is $X \subseteq B^c \cup_{i=1}^{\infty} (O_{\beta_i})$, therefor B⊆ $\cup_{i=1}^{\infty} (O_{\beta_i})$ implies that $\{O_{\beta i}\}_{i=1}^{\infty}$is a countable subcover of B■

Theorem3. The image of a p'- lindelöf space under p'-continuous map is p'- lindelöf.

Proof. Let $f: X \longrightarrow Y$ be p'-continuous mapping between (X, τ_1, σ_1) and (Y, τ_2, σ_2) with p'- lindelöf X and let $\{O_\beta\}_{\beta\in I}$ be any p'open cover for$f(X)$**,** then $X \subseteq f^{-1}$ $(\cup_{\beta\in I} O_{\ \alpha})$= $\cup_{\alpha\in I} f^{-1}(O_{\ \beta})$, then ,$\{f^{-1}(O_\beta)\}_{\beta\in I}$ is p'-open cover of X admits a countable subcover $\{ f^{-1}(O_{\beta i})\}_{i=1}^{\infty}\}$. So ,$X \subseteq f^{-1}(O_{\beta i})$ implies $f(X) \subseteq (\cup_{i=1}^{\infty} O_{\beta i})$, proves the p'- lindelofness of $f(X)$.■

Theorem4. A bitopological space X is p'-normal, if it is both p'-lindelöf and p'-regular.

Proof. Assume that A, B be τσ-closed sets in p'-regular X which holds the p'-lindelöfness property then $\forall a \in A$, there exists a τσ-open set O_β containing a such that $\bar{O}_\beta^{\tau\sigma} \cap \mathrm{B} = \phi$.

Similarity, $\forall\ b \in B$, choose τσ-open set W_b consisting b such that $\overline{\mathrm{W}}_a{}^{\tau\sigma} \cap \mathrm{A} = \phi$. Then $\{O_\beta : a \in A\}$ and $\{W_b : b \in B\}$ are p'-open cover for A, B respectively, then A, B are both p'-lindelöf by Theorem 2.

Now, define τσ-open sets S_n, T_n as the following:

$S_1 = \boldsymbol{O_{a_1}}$ $\qquad$ $T_1 = \boldsymbol{O_{b_1}} / \overline{\boldsymbol{S_1}}^{\tau\sigma}$

$S_2 = O_{a_2} / \overline{T_1}^{\tau\sigma}$ $\qquad$ $T_2 = \boldsymbol{O_{b_1}} / (\overline{\boldsymbol{S_1} \cup \boldsymbol{S_2}})^{\tau\sigma}$

$S_n = O_{a_n} / (\overline{\bigcup_{\iota=1}^{n-1} T_\iota})^{\tau\sigma}$ $\qquad$ $\boldsymbol{T_n} = \boldsymbol{O_{b_n}} / (\overline{\bigcup_{\iota=1}^{n} \boldsymbol{S_\iota}})^{\tau\sigma}$

Put $S = \bigcup_{i=1}^{\infty} S_n$ $\qquad$ and $\qquad$ $\boldsymbol{T} = \bigcup_{i=1}^{\infty} \boldsymbol{T_n}$

Note that: S, T are both τσ-open sets contain A, B respectively. Suppose to contrary that S∩T$\neq \phi$, then there exist k, m such that $S_k \cap T_m \neq \phi$. Consequently, the argument splits into two cases:

Firstly: If $k > m$ by definition $S_k = \cup_{a_k} / (\overline{\bigcup_{\iota=1}^{k-1} T_\iota})^{\tau\sigma}$ then $S \cap T = \phi$.

Secondly: If $k \leq m$ by definition $T_m = \mathrm{U}_{b_m} / (\overline{\bigcup_{\iota=1}^{m} S_\iota})^{\tau\sigma}$ then S∩T$= \phi$.

Both cases are contradict the assumption. Which establish the p'- normality of X. ■

Coroallary2. A space (X, τ, σ) is a p'-T_4 if it is three p'-regular, p'-lindelöf and p'-T_1.

Proof. By Theorem 4 and p'-normal definition.■

Theorem5. Let $\tau \leq \sigma$ be topologies on X. If (X, σ) is lindelöf, then the triple (X, τ, σ) is a p'-lindelöf space.

Proof. Suppose $\{O_\beta\}_{\beta \in I}$ be a p'-open cover of $(X, \tau, \sigma), \tau \leq \sigma$ implies that, each open set in τ, is also an open in σ. Hence, $\{O_\beta\}_{\beta \in I}$ forms an open cover of (X, σ), which admits a countable subcover $\{O_{\beta i}\}_{i=1}^{\infty}$. ■

Theorem6. The union of countably many p'-lindelöf spaces is itself p'-lindelöf.

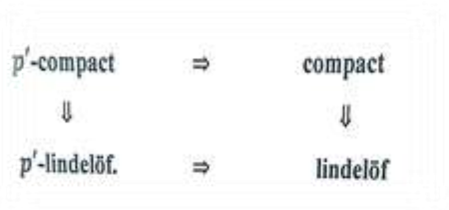
Proof. Assume $X = \bigcup_{n=1}^{\infty} X_n$ where X_n is p'-lindelöf for each n, and let be $\{O_\beta\}_{\beta \in I}$ a family of p'- open sets covering X, because each X_n is p'-lindelöf, then $\{O_\beta\}_{\beta \in I}$ can be reduced to a countable subcollection $\{O_{\beta_i}\}_{i=1}^{\infty}$ s.t $X_n \subseteq \bigcup_{i=1}^{\infty} O_{\beta_i}$.

Set the union of these countable collection as $U = \bigcup_{i=1}^{\infty} O_{\beta_i}$. Therefore, U is a countable set, and $X = \bigcup_{n=1}^{\infty} X_n \subseteq \cup O$ completes the prove.

Conclusion

This paper has explored the characterization of lindelöfness, within the sitting of pairwise topological space. By adopting the stronger framework of open set introduced in [7], we have established a discussion on the relationship between lindelöf, p'-lindelöf, compact, p'-compact spaces. Consequently, numerous results provided a deeper understanding of how these properties intersect with separation axiom in bitopology under the stronger $\tau\sigma$-open framework, highlighting several conditions under which p'-compactness implies p'-lindelöfness and other compact properties.

The following diagram summarizes the main result of this paper, established in previous section:



Furthermore, the paper also studied the conditions under which p'-lindelöf space is $p' - T_4$; such a bispace must also be p'- T_1, and p'- regular.

Future research could further investigate the behavior and underlying characteristics of lindelöfness, particularly in relation to its structural properties and preservation under various topological constructions.

In addition, potential extension and application of lindelöfness may be explored within pointfree and fuzzy topological context, where the concept could be provide new insights into generalized compactness and continuity

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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