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A Study on Subclasses of Harmonic Univalent Functions

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الدراسة حول الفئات الفرعية للدوال التوافقية أحادية التكافؤ

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Abstract:
Harmonic univalent functions constitute an essential branch of the theory of geometric functions, extending the classical theory of analytic functions. Recent studies have focused on constructing new subclasses of harmonic mappings through operator-based approaches, which allow a systematic analysis of their geometric behavior. In this work, we obtain sufficient criteria of the harmonic function classes $\mathcal{S}_H^k(\mathrm{m},\delta,\beta,\lambda,\alpha)$ and $\mathcal{C}_H^k(\mathrm{m},\delta,\beta,\lambda,\alpha)$ corresponding to starlike and convex harmonic mappings associated with k-symmetric points. Moreover, necessary conditions characterizing the membership of a function f in the subclasses $\mathcal{TS}_H^k(\mathrm{m},\delta,\beta,\lambda,\alpha)$ and $\mathcal{TC}_H^k(\mathrm{m},\delta,\beta,\lambda,\alpha)$ are established. Finally, explicit growth inequalities are obtained for functions in $\mathcal{TS}_H^k(m,\delta,\beta,\lambda,\alpha)$.

Keywords: harmonic functions, derivative operator, starlike harmonic functions, convex harmonic functions, k-symmetric points.

الملخص
تُعد الدوال التوافقية أحادية التكافؤ فرعًا أساسيًا من نظرية الدوال الهندسية، حيث تمثل امتدادًا طبيعيًا للنظرية الكلاسيكية للدوال التحليلية. وقد ركزت الدراسات الحديثة على بناء فئات جديدة من التطبيقات التوافقية من خلال مناهج تعتمد على المؤثرات، مما يسمح بدراسة منهجية لخواصها الهندسية. في هذا العمل، تم التوصل إلى شروطٍ كافية للفئات المختلفة من الدوال التوافقية $\mathcal{S}_H^k(\mathrm{m},\delta,\beta,\lambda,\alpha)$ و $\mathcal{C}_H^k(\mathrm{m},\delta,\beta,\lambda,\alpha)$ المقابلة للتطبيقات التوافقية النجمية والمحدبة المرتبطة بنقاط متناظرة من الرتبة k . وعلاوة على ذلك، يتم إثبات شروط ضرورية تميّز انتماء دالة توافقية إلى الفئتين الجزئيتين $\mathcal{TS}_H^k(\mathrm{m},\delta,\beta,\lambda,\alpha)$ و $\mathcal{TC}_H^k(\mathrm{m},\delta,\beta,\lambda,\alpha)$. وأخيرًا، نحصل على متباينات نمو صريحة للدوال التي تنتمي إلى الفئة $\mathcal{TS}_H^k(m,\delta,\beta,\lambda,\alpha)$.

الكلمات المفتاحية: الدوال التوافقية، مؤثر التفاضل، الدوال التوافقية النجمية، الدوال التوافقية المحدبة، النقاط المتناظرة من الرتبة k .

Introduction

Let $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ be the open unit disk and let $\mathcal{S}_H$ denote the class of all complex valued, harmonic, sense-preserving, univalent functions f in $\mathbb{U}$ normalized by $f(0)=h(0)=f'(0)-1=0$ and expressed as $f(z)=h(z)+\overline{g(z)}$ where h and g belong to the class $\mathcal{A}$ of all analytic functions in $\mathbb{U}$ take the form

$$h(z)=z+\sum_{n=2}^{\infty}a_nz^n, \qquad and \qquad g(z)=\sum_{n=1}^{\infty}b_nz^n. \tag{1.1}$$

As shown by Clunie and Sheil-Small [3], compactness does not hold for the class $\mathcal{S}_H$, and local univalence with preservation of sense in any simply connected domain $\mathbb{U}$ occurs whenever $|h'(z)|>|g'(z)|$.

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the subclasses $\mathcal{S}_H^*(\alpha)$ and $\mathcal{C}_H(\alpha)$ of $\mathcal{S}_H$ correspond to starlike and convex harmonic functions of order α $(0 \leq \alpha < 1)$, respectively, as introduced by Jahangiri [7]. Analytically, we have

$$f \in \mathcal{S}_H^*(\alpha) \quad \Leftrightarrow \operatorname{Im}\{\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{f(re^{i\theta})}\} = \operatorname{Re}\{\frac{zh\prime(z)-\overline{zg\prime(z)}}{h(z)+\overline{g(z)}}\} > \alpha, \tag{1.2}$$

for $0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 0 \leq \alpha < 1$ and $z \in \mathbb{U}$.

$$f \in C_{\mathcal{H}} \quad \Leftrightarrow \operatorname{Im}\{\frac{\frac{\partial}{\partial\theta}(\frac{\partial}{\partial\theta}f(re^{i\theta}))}{\frac{\partial}{\partial\theta}f(re^{i\theta})}\}$$

$$= \operatorname{Re}\{\frac{zh\prime\prime(z)+h\prime(z)-\overline{zg\prime\prime(z)+g\prime(z)}}{h\prime(z)-\overline{g\prime(z)}}\} > \alpha, \tag{1.3}$$

for $0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 0 \leq \alpha < 1$ and $z \in \mathbb{U}$.

A generalized operator of differentiation for $f \in \mathcal{A}$ was defined by Darus and Ibrahim [4] and is represented by $\mathcal{D}_{\delta,\beta,\lambda}^m$ as follows:

$$\mathcal{D}_{\delta,\beta,\lambda}^m f(z) = z + \sum_{n=2}^{\infty} [\beta(n-1)(\lambda-\delta)+1]^m a_n z^n, \tag{1.4}$$

where $\beta > 0,\ \lambda > 0, \delta \geq 0,\ \delta \neq \lambda$, and $m \in \mathbb{N}_0$.

According to Sakaguchi [9], the class S_S^* contains all functions $f \in \mathcal{A}$ that are starlike with respect to symmetric points and verify the following inequality:

$$\operatorname{Re}\{\frac{zf\prime(z)}{f(z)-f(-z)}\} > 0, \qquad (z \in \mathbb{U}). \tag{1.5}$$

The Sakaguchi class and its subclasses have been studied by different authors. Chand and Singh [2] investigated the class $S_S^{*(k)}$, consisting of functions that are starlike with respect to k-symmetric points and satisfy the following:

$$\operatorname{Re}\{\frac{zf\prime(z)}{f_k(z)}\} > 0, \qquad (z \in \mathbb{U}), \tag{1.6}$$

where

$$f_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^\nu z), \qquad (k \geq 1;\ \varepsilon = \exp(2\pi i/k)). \tag{1.7}$$

According to the definition of f_k, we have

$$f_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^\nu z)$$

$$= \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu}[\varepsilon^\nu z + \sum_{n=2}^{\infty} a_n(\varepsilon^\nu z)^n]$$

$$= z + \sum_{n=2}^{\infty} a_n \Phi_n z^n, \tag{1.8}$$

for $k \geq 1, \varepsilon = \exp(2\pi i/k), z \in \mathbb{U}$ and Φ_n given by:

$$\Phi_n = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu} = \begin{cases} 0, & n \neq lk+1, \\ 1, & n = lk+1. \end{cases} \tag{1.9}$$

Note that $S_S^{*(2)} \equiv S_S^*$.

Now, let $\mathcal{S}_H^k(m, \delta, \beta, \lambda, \alpha)$ symbolizes the class of complex-valued, sense-preserving, harmonic univalent functions $f = h + \overline{g}$ of the form (1.1), which satisfy the condition:

$$\operatorname{Im}\{\frac{\frac{\partial}{\partial\theta}\mathcal{D}_{\delta,\beta,\lambda}^m f(re^{i\theta})}{\mathcal{D}_{\delta,\beta,\lambda}^m f_k(re^{i\theta})}\} \geq \alpha \qquad (z = re^{i\theta}, z \in \mathbb{U}). \tag{1.10}$$

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Where $z = re^{i\theta}, 0 \le \theta < 2\pi, 0 \le r < 1,\ \ 0 \le \alpha < 1,\ \beta > 0,\ \lambda > 0,\ \delta \ge 0,\ \delta \ne \lambda, m \in \mathbb{N}_0,\ f_k(z) = h_k + \overline{g_k}$, $k \ge 1$ and h_k, g_k defined as

$$h_k(z) = z + \sum_{n=2}^{\infty} [\beta(n-1)(\lambda-\delta)+1]^m a_n \Phi_n z^n,$$

$$g_k(z) = \sum_{n=1}^{\infty} [\beta(n-1)(\lambda-\delta)+1]^m b_n \Phi_n z^n \quad |b_1| < 1, \tag{1.11}$$

where Φ_n given by (1.9).

Furthermore, let $\mathcal{C}_H^k(m,\delta,\beta,\lambda,\alpha)$ represents the class of sense-preserving harmonic univalent functions $f = h + \overline{g}$ of the form (1.1), satisfying the condition:

$$\operatorname{Im}\left\{\frac{\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\mathcal{D}_{\delta,\beta,\lambda}^{m} f(re^{i\theta})\right)}{\frac{\partial}{\partial\theta}\mathcal{D}_{\delta,\beta,\lambda}^{m} f_k(re^{i\theta})}\right\} \ge \alpha, \tag{1.12}$$

with, $z = re^{i\theta}, 0 \le \theta < 2\pi, 0 \le r < 1,\ 0 \le \alpha < 1, \delta \ge 0,\ \beta > 0,\ \lambda > 0,\ \delta \ne \lambda, m \in \mathbb{N}_0,\ f_k(z) = h_k + \overline{g_k}$ $(k \ge 1)$ and h_k, g_k given by (1.11).

Moreover, let $\mathcal{TS}_H^k(m,\delta,\beta,\lambda,\alpha)$ and $\mathcal{TC}_H^k(m,\delta,\beta,\lambda,\alpha)$ represent the subclasses of $\mathcal{S}_H^k(m,\delta,\beta,\lambda,\alpha)$ and $\mathcal{C}_H^k(m,\delta,\beta,\lambda,\alpha)$, respectively for which h and g in $f = h + \overline{g}$, take the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n,$$

$$g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \ \ (|b_1| < 1), \tag{1.13}$$

h_k and g_k belong to $f_k = h_k + \overline{g_k}$, are given by

$$h_k(z) = z - \sum_{n=2}^{\infty} [\beta(n-1)(\lambda-\delta)+1]^m |a_n| \Phi_n z^n,$$

$$g_k(z) = \sum_{n=1}^{\infty} [\beta(n-1)(\lambda-\delta)+1]^m |b_n| \Phi_n z^n, \tag{1.14}$$

where Φ_n given by (1.9).

It is clear that the classes:

$\mathcal{S}_H^1(0,\delta,\beta,\lambda,0) \equiv \mathcal{S}_H^*,\ \mathcal{TS}_H^1((0,\delta,\beta,\lambda,0) \equiv \mathcal{TS}^*,\ \mathcal{C}_H^1(0,\delta,\beta,\lambda,0) \equiv \mathcal{C}_H,\ \mathcal{TC}_H^1(0,\delta,\beta,\lambda,0) \equiv \mathcal{TC}_H$, were studied by Silverman [10].

$\mathcal{S}_H^1(0,\delta,\beta,\lambda,\alpha) \equiv \mathcal{S}_H^*(\alpha),\ \mathcal{TS}_H^1(0,\delta,\beta,\lambda,\alpha) \equiv \mathcal{TS}^*(\alpha),\ \mathcal{C}_H^1(0,\delta,\beta,\lambda,\alpha) \equiv \mathcal{C}_H(\alpha), \mathcal{TC}_H^1(0,\delta,\beta,\lambda,\alpha) \equiv \mathcal{TC}_H(\alpha)$ were studied by Jahangiri [7].

$\mathcal{S}_H^2(0,\delta,\beta,\lambda,\alpha) \equiv \mathcal{S}_{HS}^*(\alpha)$ and $\mathcal{TS}_H^2(0,\delta,\beta,\lambda,\alpha) \equiv \mathcal{TS}_{HS}^*(\alpha)$ were studied by Ahuja and Jahangiri [1] and G̈uney [6], respectively.

To derive our main results, Theorems 1.1 and 1.2 to Jahangiri [7] are needed.

Theorem 1.1 *[7]. Let* $f = h + \overline{g} \in \mathcal{S}_H$, *if*

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} |b_n| \le 1, \tag{1.15}$$

where $0 \le \alpha < 1$, then f is harmonic, sense-preserving, univalent in $\mathbb{U}$, and f is starlike harmonic of order α denoted by $\mathcal{S}_H(\alpha)$. Condition (1.15) is also necessary if $f \in \mathcal{TS}_H(\alpha)$.

Theorem 1.2 *[10]. Let* $f = h + \overline{g} \in \mathcal{S}_H$, *if*

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{(1-\alpha)} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{(1-\alpha)} |b_n| \le 1, \tag{1.16}$$

where $0 \le \alpha < 1$, then the function f is harmonic and univalent in $\mathbb{U}$ with preservation of orientation, and hence f lies in the class $\mathcal{C}_H(\alpha)$ of convex harmonic functions of order α. Condition (1.16) is required for every $f \in \mathcal{TC}_H(\alpha)$.

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Main results

First, the significant conclusions regarding the class $\mathcal{S}_H^k(m,\delta,\beta,\lambda,\alpha)$ will be presented.

Theorem 2.1. *For $f\in\mathcal{S}_H^k(m,\delta,\beta,\lambda,\alpha)$, where $f=h+\overline{g}$, h and g defined by (1.1), then $f_k(z)=h_k+\overline{g_k}$, h_k and g_k given by (1.11) belongs to $\mathcal{S}_H^k(m,\delta,\beta,\lambda,\alpha)$.*

Proof. Suppose that $f\in\mathcal{S}_H^k(m,\delta,\beta,\lambda,\alpha)$. By substituting $re^{i\theta}$ by $\varepsilon^\nu re^{i\theta}$, $(\nu=0,1,2,\dots,k-1;\ \varepsilon^k=1)$ in (1.10) respectively, we observe that (1.10) is also true such that

$$\operatorname{Im}\{\frac{\frac{\partial}{\partial\theta}\mathcal{D}^m_{\delta,\beta,\lambda}f(\varepsilon^\nu re^{i\theta})}{\mathcal{D}^m_{\delta,\beta,\lambda}f_k(\varepsilon^\nu re^{i\theta})}\}\geq\alpha,\qquad(\nu=0,1,2,\dots,k-1).\tag{2.1}$$

Based on the definition of $f_k(z)$ and $\varepsilon^k=1$, we know $f_k(\varepsilon^\nu re^{i\theta})=\varepsilon^\nu f_k(re^{i\theta})$. Let $\nu=0,1,2,\dots,k-1$ in (2.1) respectively, and hence their sum is

$$\operatorname{Im}\{\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{\frac{\partial}{\partial\theta}\mathcal{D}^m_{\delta,\beta,\lambda}f(\varepsilon^\nu re^{i\theta})}{\varepsilon^\nu\mathcal{D}^m_{\delta,\beta,\lambda}f_k(re^{i\theta})}\}=\operatorname{Im}\{\frac{\frac{\partial}{\partial\theta}\mathcal{D}^m_{\delta,\beta,\lambda}f_k(re^{i\theta})}{\mathcal{D}^m_{\delta,\beta,\lambda}f_k(re^{i\theta})}\}\geq\alpha,\tag{2.2}$$

here $f_k(z)$ belongs to $\mathcal{S}_H^k(m,\delta,\beta,\lambda,\alpha)$.

Furthermore, a sufficient criteria for harmonic functions in $\mathcal{S}_H^k(m,\delta,\beta,\lambda,\alpha)$ is obtained.

Theorem 2.2 *Let h and g given by (1.1), $f=h+\overline{g}$ and h_k, g_k given by (1.12), $f_k=h_k+\overline{g_k}$. If*

$$\sum_{n=1}^{\infty}[\frac{n-\alpha\Phi_n}{1-\alpha}|a_n|+\frac{n+\alpha\Phi_n}{1-\alpha}|b_n|][\beta(n-1)(\lambda-\delta)+1]^m\leq 2,\tag{2.3}$$

where $0\leq\alpha\leq1$, $\beta>0$, $\delta\geq0,\lambda>0$, $\delta\neq\lambda, m\in\mathbb{N}_0, a_1=\Phi_1=1$, and Φ_n *given by* (1.9). *Then f is harmonic univalent in $\mathbb{U}$, with sense-preserving and $f\in\mathcal{S}_H^k(m,\delta,\beta,\lambda,\alpha)$.*

Proof. For $a_1=1$ in Theorem 1.1, then

$$\sum_{n=1}^{\infty}[\frac{n-\alpha}{1-\alpha}|a_n|+\frac{n+\alpha}{1-\alpha}|b_n|]$$

$$\leq\sum_{n=1}^{\infty}[\frac{n-\alpha\Phi_n}{1-\alpha}|a_n|+\frac{n+\alpha\Phi_n}{1-\alpha}|b_n|][\beta(n-1)(\lambda-\delta)+1]^m$$

$$\leq 2,$$

Using Theorem 1.1, we have that f is starlike and harmonic univalent in $\mathbb{U}$, with sense-preserving. To show $f\in\mathcal{S}_H^k(m,\delta,\beta,\lambda,\alpha)$, the condition (1.10) is needed to obtain that:

$$\operatorname{Im}\{\frac{\frac{\partial}{\partial\theta}\mathcal{D}^m_{\delta,\beta,\lambda}f(re^{i\theta})}{\mathcal{D}^m_{\delta,\beta,\lambda}f_k(re^{i\theta})}\}=\operatorname{Re}\{\frac{z(\mathcal{D}^m_{\delta,\beta,\lambda}h(z))'-\overline{z(\mathcal{D}^m_{\delta,\beta,\lambda}g(z))'}}{\mathcal{D}^m_{\delta,\beta,\lambda}h_k(z)+\overline{\mathcal{D}^m_{\delta,\beta,\lambda}g_k(z)}}\}$$

$$=\operatorname{Re}\left\{\frac{A(z)}{B(z)}\right\}$$

$$\geq\alpha,$$

where $z=re^{i\theta}, 0\leq r<1, 0\leq\theta<2\pi$, $0\leq\alpha<1,\beta>0$, $\lambda>0,\delta\geq0$, $\delta\neq\lambda, m\in\mathbb{N}_0$ and $k\geq1$.

$$A(z)=z(\mathcal{D}^m_{\delta,\beta,\lambda}h(z))'-\overline{z(\mathcal{D}^m_{\delta,\beta,\lambda}g(z))'}$$

$$=z+\sum_{n=2}^{\infty}n[\beta(n-1)(\lambda-\delta)+1]^m a_n z^n-\overline{\sum_{n=1}^{\infty}n[\beta(n-1)(\lambda-\delta)+1]^m b_n z^n},\tag{2.4}$$

and

$$B(z)=\mathcal{D}^m_{\delta,\beta,\lambda}f_k(z)$$

$$=z+\sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\delta)+1]^m a_n\Phi_n z^n+\overline{\sum_{n=1}^{\infty}[\beta(n-1)(\lambda-\delta)+1]^m b_n\Phi_n z^n}\tag{2.5}$$

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where Φ_n given by (1.9). By using $\text{Re}(w) \geq \alpha \iff |1-\alpha+w| \geq |1+\alpha-w|$, it enough to see that

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)| \geq 0. \tag{2.6}$$

By replacing $A(z)$ and $B(z)$ as given in (2.4) and (2.5) in (2.6), we get

$$\begin{aligned}
&|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)| \\
&\quad = |(1-\alpha)\mathcal{D}^m_{\delta,\beta,\lambda}h_k(z)+z(\mathcal{D}^m_{\delta,\beta,\lambda}h(z))' + \overline{(1-\alpha)\mathcal{D}^m_{\delta,\beta,\lambda}g_k(z)-z(\mathcal{D}^m_{\delta,\beta,\lambda}g(z))'}| \\
&\quad -|(1+\alpha)\mathcal{D}^m_{\delta,\beta,\lambda}h_k(z)-z(\mathcal{D}^m_{\delta,\beta,\lambda}h(z))' + \overline{(1+\alpha)\mathcal{D}^m_{\delta,\beta,\lambda}g_k(z)+z(\mathcal{D}^m_{\delta,\beta,\lambda}g(z))'}|
\end{aligned}$$

$$= |(2-\alpha)z+\sum_{n=2}^{\infty}(n+(1-\alpha)\Phi_n)[\beta(n-1)(\lambda-\delta)+1]^m a_n z^n$$

$$-\overline{\sum_{n=1}^{\infty}(n-(1-\alpha)\Phi_n)[\beta(n-1)(\lambda-\delta)+1]^m b_n z^n}|$$

$$-|-\alpha z+\sum_{n=2}^{\infty}(n-(1+\alpha)\Phi_n)[\beta(n-1)(\lambda-\delta)+1]^m a_n z^n$$

$$-\overline{\sum_{n=1}^{\infty}(n+(1+\alpha)\Phi_n)[\beta(n-1)(\lambda-\delta)+1]^m b_n z^n}|$$

$$\geq (2-\alpha)|z|-\sum_{n=2}^{\infty}(n+(1-\alpha)\Phi_n)[\beta(n-1)(\lambda-\delta)+1]^m|a_n||z|^n$$

$$-\sum_{n=1}^{\infty}(n-(1-\alpha)\Phi_n)[\beta(n-1)(\lambda-\delta)+1]^m|b_n||z|^n$$

$$-\alpha|z|-\sum_{n=2}^{\infty}(n-(1+\alpha)\Phi_n)[\beta(n-1)(\lambda-\delta)+1]^m|a_n||z|^n$$

$$-\sum_{n=1}^{\infty}(n+(1+\alpha)\Phi_n)[\beta(n-1)(\lambda-\delta)+1]^m|b_n||z|^n$$

$$= 2(1-\alpha)|z|\{1-\sum_{n=2}^{\infty}\frac{n-\alpha\Phi_n}{1-\alpha}[\beta(n-1)(\lambda-\delta)+1]^m|a_n||z|^{n-1}$$

$$-\sum_{n=1}^{\infty}\frac{n+\alpha\Phi_n}{1-\alpha}[\beta(n-1)(\lambda-\delta)+1]^m|b_n||z|^{n-1}\}$$

$$\geq 2(1-\alpha)|z|\{1-\sum_{n=2}^{\infty}\frac{n-\alpha\Phi_n}{1-\alpha}[\beta(n-1)(\lambda-\delta)+1]^m|a_n|$$

$$-\sum_{n=1}^{\infty}\frac{n+\alpha\Phi_n}{1-\alpha}[\beta(n-1)(\lambda-\delta)+1]^m|b_n|\}$$

$$\geq 2(1-\alpha)\{1-\sum_{n=2}^{\infty}\frac{n-\alpha\Phi_n}{1-\alpha}[\beta(n-1)(\lambda-\delta)+1]^m|a_n|$$

$$-\sum_{n=1}^{\infty}\frac{n+\alpha\Phi_n}{1-\alpha}[\beta(n-1)(\lambda-\delta)+1]^m|b_n|\} \geq 0,$$

By (2.3), the proof is complete.

Next, the necessity of condition (2.3) for functions in $\mathcal{TS}^k_H(m,\delta,\beta,\lambda,\alpha)$, is established.

Theorem 2.3 *Let h and g are given by (1.13), $f = h+\overline{g}$ and $f_k = h_k+\overline{g_k}$ with h_k and g_k are given by (1.14). Then $f \in \mathcal{TS}^k_H(m,\delta,\beta,\lambda,\alpha) \iff$ (2.3) holds.*

Proof. The *if* part follows from Theorem 2.2 noting that if the analytic and co-analytic parts of $f = h+\overline{g} \in \mathcal{S}^k_H(m,\delta,\beta,\lambda,\alpha)$ are of the form (1.13) then $f \in \mathcal{TS}^k_H(m,\delta,\beta,\lambda,\alpha)$. For the *only if* part, we show that $f \notin \mathcal{TS}^k_H(m,\delta,\beta,\lambda,\alpha)$ if the condition (2.3) does not hold. Note that a necessary and sufficient condition for $f = h+\overline{g}$ given by (1.13) to be in $\mathcal{S}^k_H(m,\delta,\beta,\lambda,\alpha)$, is that

$$\text{Re}\{\frac{z(\mathcal{D}^m_{\delta,\beta,\lambda}h(z))'-\overline{z(\mathcal{D}^m_{\delta,\beta,\lambda}g(z))'}}{\mathcal{D}^m_{\delta,\beta,\lambda}h_k(z)+\overline{\mathcal{D}^m_{\delta,\beta,\lambda}g_k(z)}}\} \geq \alpha.$$

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This is equivalent to

$$\text{Re}\{\frac{z(\mathcal{D}^m_{\delta,\beta,\lambda}h(z))' - \overline{z(\mathcal{D}^m_{\delta,\beta,\lambda}g(z))'}}{\mathcal{D}^m_{\delta,\beta,\lambda}h_k(z) + \overline{\mathcal{D}^m_{\delta,\beta,\lambda}g_k(z)}}\} - \alpha \geq 0.$$

$$= \Re\{\frac{(1-\alpha)z - \sum_{n=2}^{\infty}(n-\alpha\Phi_n)[\beta(n-1)(\lambda-\delta)+1]^m|a_n|z^n}{z - \sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\delta)+1]^m\Phi_n|a_n|z^n + \overline{\sum_{n=1}^{\infty}[\beta(n-1)(\lambda-\delta)+1]^m\Phi_n|b_n|z^n}}$$

$$- \frac{\overline{\sum_{n=1}^{\infty}(n+\alpha\Phi_n)[\beta(n-1)(\lambda-\delta)+1]^m|b_n|z^n}}{z - \sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\delta)+1]^m\Phi_n|a_n|z^n + \overline{\sum_{n=1}^{\infty}[\beta(n-1)(\lambda-\delta)+1]^m\Phi_n|b_n|z^n}}\}$$

$$\leq |\frac{(1-\alpha)z - \sum_{n=2}^{\infty}(n-\alpha\Phi_n)[\beta(n-1)(\lambda-\delta)+1]^m|a_n|z^n}{z - \sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\delta)+1]^m\Phi_n|a_n|z^n + \sum_{n=1}^{\infty}[\beta(n-1)(\lambda-\delta)+1]^m\Phi_n|b_n|\overline{z^n}}$$

$$- \frac{\sum_{n=1}^{\infty}(n+\alpha\Phi_n)[\beta(n-1)(\lambda-\delta)+1]^m|b_n|\overline{z^n}}{z - \sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\delta)+1]^m\Phi_n|a_n|z^n + \sum_{n=1}^{\infty}[\beta(n-1)(\lambda-\delta)+1]^m\Phi_n|b_n|\overline{z^n}}|$$

Suppose that condition (2.3) is not satisfied, then

$$\leq \frac{(1-\alpha) - (\sum_{n=2}^{\infty}(n-\alpha\Phi_n)|a_n| + \sum_{n=1}^{\infty}(n+\alpha\Phi_n)|b_n|)[\beta(n-1)(\lambda-\delta)+1]^m}{1 - (\sum_{n=2}^{\infty}\Phi_n|a_n| - \sum_{n=1}^{\infty}\Phi_n|b_n|)[\beta(n-1)(\lambda-\delta)+1]^m},$$

that is,

$$(1-\alpha) - (\sum_{n=2}^{\infty}(n-\alpha\Phi_n)|a_n| + \sum_{n=1}^{\infty}(n+\alpha\Phi_n)|b_n|)[\beta(n-1)(\lambda-\delta)+1]^m \leq 0.$$

Hence $f \notin \mathcal{TS}_H^k(m,\delta,\beta,\lambda,\alpha)$. Thus, the proof is concluded.

The following theorem presents the growth result.

Theorem 2.4 *If $f \in \mathcal{TS}_H^k(m,\delta,\beta,\lambda,\alpha)$, then*

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{[\beta(\lambda-\delta)+1]^m}(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|)r^2, \qquad |z| = r < 1,$$

$$|f(z)| \geq (1-|b_1|)r - \frac{1}{[\beta(\lambda-\delta)+1]^m}(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|)r^2, \qquad |z| = r < 1.$$

Proof. it suffices to prove the second inequality, as the first can be shown by a similar argument. Let $f \in \mathcal{TS}_H^k(m,\delta,\beta,\lambda,\alpha)$. Taking the modulus of $f(z)$ we obtain

$$|f(z)| \geq (1-|b_1|)r - \sum_{n=2}^{\infty}[|a_n| + |b_n|]r^n$$

$$\geq (1-|b_1|)r - \sum_{n=2}^{\infty}[|a_n| + |b_n|]r^2$$

$$\geq (1-|b_1|)r - \frac{1-\alpha}{(2-\alpha\Phi_2)[\beta(\lambda-\delta)+1]^m}\sum_{n=2}^{\infty}\frac{2-\alpha\Phi_2}{1-\alpha}[|a_n| + |b_n|][\beta(\lambda-\delta)+1]^m r^2$$

$$\geq (1-|b_1|)r - \frac{1-\alpha}{(2-\alpha)[\beta(\lambda-\delta)+1]^m}\sum_{n=2}^{\infty}\left(\frac{n-\alpha\Phi_n}{1-\alpha}|a_n| + \frac{n+\alpha\Phi_n}{1-\alpha}|b_n|\right)[\beta(\lambda-\delta)+1]^m r^2$$

$$\geq (1-|b_1|)r - \frac{1-\alpha}{(2-\alpha)[\beta(\lambda-\delta)+1]^m}(1 - \frac{1+\alpha}{1-\alpha}|b_1|)r^2 \qquad \text{by(2.3)}$$

$$= (1-|b_1|)r - \frac{1}{[\beta(\lambda-\delta)+1]^m}(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|)r^2.$$

The following covering result is derived from the second inequality in Theorem 2.4.

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Corollary 2.5 *If* $f \in \mathcal{T}S_H^k(m,\delta,\beta,\lambda,\alpha)$ *then*

$$\{w:|w|<1-\frac{(1-\alpha)}{(2-\alpha)[\beta(\lambda-\delta)+1]^m}-(1-\frac{(1+\alpha)}{(2-\alpha)[\beta(\lambda-\delta)+1]^m})|b_1|\}\subset f(\mathbb{U}).$$

The convex case

The method for proving the following theorems in the convex case mirrors that of Theorems 2.1, 2.2 and 2.3 for the starlike case, substituting Theorem 1.2 for Theorem 1.1.

Theorem 3.1 *For* $f \in \mathcal{C}_H^k(m,\delta,\beta,\lambda,\alpha)$ *where* $f = h+\overline{g}$ *with* h *and* g *are represented by (1.1), then* $f_k(z) = h_k+\overline{g_k}$ *with* h_k *and* g_k *are defined by (1.11) is in* $\mathcal{C}_H^k(m,\delta,\beta,\lambda,\alpha)$.

Theorem3.2 *For* $f = h+\overline{g}$ *where* h *and* g *are defined by (1.1) and* $f_k = h_k+\overline{g_k}$, h_k *and* g_k *are represented by (1.12). If*

$$\sum_{n=1}^{\infty}[\frac{n(n-\alpha\Phi_n)}{1-\alpha}|a_n|+\frac{n(n+\alpha\Phi_n)}{1-\alpha}|b_n|][\beta(n-1)(\lambda-\delta)+1]^m\leq 2,$$

where $a_1=\Phi_1=1,\ 0\leq\alpha\leq 1, \delta\geq 0,\ \beta>0,\ \lambda>0,\ \delta\neq\lambda, m\in\mathbb{N}_0$ and Φ_n given by (1.9). Hence f is harmonic univalent function and sense-preserving in the open unit disk and $f\in\mathcal{C}_H^k(m,\delta,\beta,\lambda,\alpha)$.

Theorem 3.3 *For* $f = h+\overline{g}$ *with* h *and* g *are given by (1.13) and* $f_k = h_k+\overline{g_k}$ *with* h_k *and* g_k *are expressed as in (1.14). Then* f *in* $\mathcal{T}C_H^k(m,\delta,\beta,\lambda,\alpha)\Leftrightarrow$

$$\sum_{n=1}^{\infty}[\frac{n(n-\alpha\Phi_n)}{(1-\alpha)}|a_n|+\frac{n(n+\alpha\Phi_n)}{(1-\alpha)}|b_n|][\beta(n-1)(\lambda-\delta)+1]^m\leq 2,$$

where $a_1=\Phi_1=1,\ 0\leq\alpha\leq 1, \delta\geq 0,\ \beta>0,\ \lambda>0,\ \delta\neq\lambda, m\in\mathbb{N}_0$ and Φ_n given by (1.9).

Remark 2.1. *related contributions on harmonic functions are reported in [5, 8, 11].*

Conclusion

In the present work, we introduce and study new subclasses of harmonic univalent functions that are k-symmetric starlike and convex which defined by a derivative operator and investigate some of their geometric properties.

Compliance with ethical standards

Disclosure of conflict of interest

The author(s) declare that they have no conflict of interest.

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