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Development of a Higher-Order Iterative Method with Variable Parameters for the Roots of Starlike Functions

Asma A. Almahrouq [1*], Salma F. R. Naji [2], M. Darus [3]
[1] Department of Mathematics, Faculty of Science - Al-Asabaa, Gharyan University, Al-Asabaa, Libya
[2] Department of Mathematics, Faculty of Science, University of Sabratha, Libya
[3] School of Mathematical Sciences, Faculty of Science and Technology, University Kebangsaan Malaysia, 43600 UKM Bangi, Selangor DE, Malaysia

تطوير طريقة تكرارية من رتبة أعلى بمعاملات متغيرة لإيجاد جذور الدوال النجمية

أسماء المحروق [1*]، سالمة ناجي [2] ، ماسلينا داروس [3]
[1] قسم الرياضيات، كلية العلوم ـ الأصابعة، جامعة غريان، الأصابعة، ليبيا
[2] قسم الرياضيات، كلية العلوم، جامعة صبراتة، ليبيا
[3] قسم الرياضيات، كلية العلوم والتكنولوجيا، الجامعة الوطنية الماليزية، ماليزيا

[*]Corresponding author: almhrwqasma611@gmail.com

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Abstract
In this present paper, a new iterative scheme is obtained, designed to compute the roots of starlike functions and to address the constrains faced by traditional iterative algorithms when dealing with this specific class of functions. This is achieved by combining the development formula with coefficients of the variable m and α, which demonstrating that both the order of convergence and computational efficiency improve as these coefficients increase. The numerical examples confirm the proposed scheme's ability to solve the drawbacks of classical methods, achieving high numerical efficiency and a convergence rate that is faster compared to known traditional methods.

Keywords: Starlike function; Iterative methods; Nonlinear equations; Order of convergence; Numerical stability.

الملخص
في هذه الورقة يتم الحصول على مخططاً تكرارياً مصمماً لحساب جذور الدوال النجمية ومعالجة القيود التي تواجه الخوارزميات التكرارية التقليدية عند التعامل مع هذه الفئة المحددة من الدوال. وذلك بدمج طريقة الصيغة المطورة بمعاملات متغيرة α, m مما يوضح ان كلاً من رتبة التقارب والكفاءة الحسابية يتم تحسينهما مع زيادة هذه المعاملات. وتؤكد الأمثلة العددية قدرة المخطط المقترح على حل العيوب الكلاسيكية محققاً كفاءة عددية فائقة ومعدل تقارب أسرع مقارنة بالطرق التقليدية المعروفة.

الكلمات المفتاحية: الدوال النجمية، الطرق التكرارية، المعادلات الغير خطية، رتبة التقارب، الاستقرار العددي.

1. Introduction

Nonlinear equations are frequently encountered in various disciplines of science and engineering, yet deriving their analytical solutions is often infeasible. In such instances, where an exact solution cannot be found, numerical iterative methods become indispensable.

While classical methods are often subject to restrictive conditions that may limit their practical scope, this work proposes a novel scheme based on the specific properties of starlike functions.

These functions are instrumental in characterizing the behavior of nonlinear problems that fail to satisfy traditional convexity requirements. Accordingly, we first define the class of functions under study.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad , z \in D$$

Which are analytic in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \ and \ |z| < 1\}$.

A function $F \in A$ is called starlike with respect to the origin if it satisfies the following condition:

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad , z \in \mathbb{D}$$

This condition ensures that the image $F(D)$ is star-shaped domain with respect to the origin. The class of univalent starlike function is typically denoted by S^*. These functions are characterized by specific growth properties and coefficient estimates making them a pivotal tool in geometric function theory.

Despite their regular geometric structure, starlike functions impose fundamental constraints on classical algorithms. Recent studies, such as Alotaibi et al. [1], have focused on uncovering the deep geometric characteristics of these functions, including distortion properties and growth rates. Furthermore, numerical investigations, such as the work of Kumar et al. [2], have revealed practical difficulties encountered by classical methods when finding the roots of such functions.

These numerical methods often show slow convergence or even failure, especially when dealing with small derivatives or clustered multiple roots. As noted in the aforementioned studies, traditional approaches fail to account for the specific geometric constraints of starlike function, treating them merely as general analytic functions. This oversight ignores important information governing root behavior and distribution. Therefore, there is a clear necessity to develop a numerical framework tailored to the unique characteristics of these functions. By focusing on this specific class, which exhibits structural rigidity in root distribution, this study presents a higher order iterative formula to improve numerical efficiency.

This approach integrates the theoretical properties of star like functions with numerical implementation as established in the following definition.

The proposed iterative

1. **Definition:** let $f(z)$ be an analytic function belonging to the class of starlike function in the open disk D. we define the iterative path for the convergence sequence $\{z_n\}_{n=0}^{\infty}$ by the following mathematical expression

$$z_{n+1} = z_n - \sum_{k=1}^{m} \alpha^{k-1} \left[\frac{f(z_n)}{f'(z_n)}\right]^k \qquad [1.1]$$

Remark 1:

Observe that for $m = 1$ and $\alpha = 1$ the proposed formula in definition [1.1] reduces to the classical Newton's method. This method is widely utilized for locating the roots of nonlinear equations with its advantages and numerical limitations extensively discussed in [3, 4,5}.

However, its direct application to starlike functions without exploiting their geometric properties often yields less accurate result than the proposed approach[6].

2. Efficiency analysis of the proposed approach:

through this framework, we identify several key advantages addressing specific scenarios computational:

1. **Mitigation of geometric distortion via the constraint $0 < \alpha \leq 1$.**

Starlike functions exhibit complex geometric mappings characterized by severe twists near their boundaries. Consequently, as indicated in [3], the efficiency of an iterative system in the complex plane depends on more than just the numerical convergence order. While traditional methods lacking control parameters often suffer from chaotic behavior and path divergence, the inclusion of the weighting factor α serves as a practical mechanism to suppress geometric distortions. This stabilizes the iterative trajectory and ensures precise convergence toward the root.

2. **Ensuring stability under vanishing derivatives**

The proposed formulation effectively addresses the phenomenon where the derivative $\acute{f}(z)$ approaches zero-a scenario that typically leads to the failure of traditional methods. By incorporating the weighting factor α^{k-1}, the formula effectively suppresses the inflation of the ratio $\left|\frac{f(z)}{f'(z)}\right|^k$. This is analytically demonstrated by reformulating the general terms as $M.(\alpha M)^{k-1}$, which maintains numerical stability even in sensitive regions of the search space.

3. Enhancement of convergence order $(m+1)$.

The proposed formula mitigates numerical stagnation and slow convergence by employing the parameter m. Through this configuration, the convergence order is formally elevated from the classical second order to a higher order of $(m+1)$, significantly reducing the iterations required for starlike functions.

3. Convergence analysis:

***Theorem 3.1*:** let $f: D \subseteq C \to C$ be a sufficiently smooth starlike function in a neighborhood of a simple zero $\alpha^* \in D$. If the initial approximation z_0 is sufficiently close to α^* ,then the method defined in [1.1] is convergent with the order of convergence $(m+1)$ provided that the damping parameter is $\alpha = 1$.

Proof:

Let $e_n = z_n - \alpha^*$ denote the error at the n-th iteration, where α^* is a simple zero of the starlike function $f(z)$. By applying the Taylor series expansion of $f(z_n)$ and $f'(z_n)$ about the zero α^*, we obtain:

$$f(z_n) = f'(\alpha^*)\,[e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots + c_{m+1} e_n^{m+1} + O(e_n^{m+2})]$$

And

$$f'(z_n) = f'(\alpha^*)\,[1 + 2c_2 e_n + 3c_3 e_n^2 + \cdots + (m+1)c_{m+1} e_n^{m} + O(e_n^{m+1})]$$

Where the constants c_j are defined as $c_j = \frac{1}{j!}\frac{f^j(\alpha)}{f'(\alpha)}$, $j \geq 2$.

From the above expansions, the ratio $\frac{f(z_n)}{f'(z_n)}$ can be expressed as

$$\frac{f(z_n)}{f'(z_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + \cdots + O(e_n^{m+1})$$

Substituting the ratio into the proposed iterative scheme [1.1], and considering the case where the damping parameter is set to $\alpha = 1$, the summation structure leads to a systematic cancellation of the error terms $e_n, e_n^2, \dots, e_n^m$.

Specifically, the cumulative effect of the weighting factor α^{k-1} and the high order variants ensures that:

$$z_{n+1} - \alpha^* = C e_n^{m+1} + O(e_n^{m+2})$$

Where C is a constant depending on the derivatives of f at α^*. Consequently, the error relation satisfies:

$$e_{n+1} \approx C\, e_n^{m+1}$$

Which demonstrates that the method attains an order of convergence of at least $m+1$.

Based on the convergence results established in Theorem 3.1, we can now analytically trace the recursive progression of the error sequence $\{e_n\}$. By observing the leading error terms derived in our proof, we provide a detailed breakdown of the error decay at each iteration stage. This allows us to demonstrate how the proposed method maintains its theoretical efficiency in practice. The following table summarizes these relationships:

TABLE 1

n	e_n
0	e_0
1	$C\, e_0^{m+1}$
2	$C^{\frac{(m+1)^2-1}{m}}\, e_0^{(m+1)^2}$
3	$C^{\frac{(m+1)^3-1}{m}}\, e_0^{(m+1)^3}$
4	$C^{\frac{(m+1)^4-1}{m}}\, e_0^{(m+1)^4}$
⋮	⋮

The recursive patterns in Table 1, verify the theoretical consistency of our approach. We observe that the error diminishes at a rate of $(m+1)^n$ in each successive stage, confirming that the integration of parameter m effectively accelerates convergence toward the root α^* of the starlike function.

4. Algorithms and Numerical Examples

In this section, we present the general algorithm required to implement the proposed iterative method [1.1] for a order m given:

step 1: For a given tolerance $\varepsilon > 0$ and maximum iteration N , choose the initial approximation z_0 and the convergence order $(m+1)$ by setting $m \geq 1$. Initialize the counter $n = 0$.

Step 2: Compute the next iterate z_{n+1} using the following summation formula:

$$z_{n+1} = z_n - \left(\alpha^0\left[\frac{f(z_n)}{f'(z_n)}\right]^1 + \alpha^1\left[\frac{f(z_n)}{f'(z_n)}\right]^2 + \cdots + \alpha^{m-1}\left[\frac{f(z_n)}{f'(z_n)}\right]^m\right) \quad .$$

Step 3: The process terminates if either $|z_{n+1} - z_n| < \varepsilon$ or $n > N$. If the stopping criterion is met, output z_{n+1} as the approximate root.
Step 4: If the criteria in step 3 are not satisfied, update the iteration index $n = n + 1$ and return step 2.

a. *Numerical example on starlike functions*

To evaluate the efficiency and robustness of the proposed algorithm, we apply Algorithm 4.1 to several test functions, beginning with the koebe function, which serves as fundamental starlike function.

Example 1: Consider the Koebe function defined by:

$$f(z) = \frac{z}{(1-z)^2} = 0$$

Using an initial approximation $z_0 = 0.2$ and $m = 3$. We compare the performance of our method against the classical Newton's method, a third order method [7] and our method [1.1].
The comparative results, including iteration counts and residual errors, are presented in the following tables.

TABLE 2

n	Original Newtons method	A third order method	Present method [1.1]
1	0.052631578947	0.013850415512	0.003462603878
2	0.004149377593	0.000008412251	0.000000010849
3	0.000025931211	0.00000001043	Converged
4	0.000000010343	converged	-

Example 2: We further test the algorithm using the exponential function $f(z) = e^z - 1 = 0$. Which has a simple root at $\alpha^* = 0$. Taking $z_0 = 1$ and $m = 4$, the numerical results are compared with other methods in Table 3:

TABLE 3

n	Original Newtons method	A third order method	Present method [1.1]
1	0.367879441171	0.187310512411	0.016301284751
2	0.058080078319	0.001948512033	0.000000000108
3	0.001651697110	0.000000002451	Converged
4	0.000001363539	Converged	

Remark 4.1: A comparative analysis of the results in Tables 2 and 3 reveals a direct correlation between the parameter m and the efficiency of the proposed method. We observe that as the value of m increases, the algorithm exhibits a significant acceleration in reaching the root with higher numerical precision. This confirms that m acts as a decisive factor in optimizing the computational cost, effectively reducing the number of iterations required for convergence across different classes of starlike function.

Conclusion

In this paper, we have investigated the iterative method defined by the expression [1.1] and developed variants of this scheme by incorporating the properties of starlike functions. It is demonstrated that the resulting method exhibit high efficiency when applied specifically to this class of functions. Furthermore, numerical results confirm that as the value of the parameter m increases, both convergence speed and accuracy improve significantly. The efficiency of these methods is highly competitive with existing high order iterations, providing a robust and effective approach for addressing challenges associated with starlike functions.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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