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On the Intersection of Subgroup Products with Non-Cyclic Abelian Normal 2-Subgroups

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حول تقاطع حواصل ضرب الزمر الجزئية مع الزمر الجزئية ثنائية الناظمية الأبيليّة غير الدورية

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Abstract:
Sufficient conditions are established to ensure that the intersection of a product of two subgroups with an abelian normal subgroup is itself a subgroup within a finite group framework. It is proven that if K is an abelian 2-group and A is a subgroup of odd order, then the commutativity condition $AB \cap K = BA \cap K$ is sufficient to guarantee that the intersection $AB \cap K$ is a subgroup, regardless of whether K is cyclic or non-cyclic. This result generalizes and extends known findings from the cyclic case to the more general non-cyclic abelian setting.

Keywords: Subgroup products, Abelian 2-groups, Coprime action, Non-cyclic sections, First cohomology group, Principal derivation.

الملخص:
تم وضع شروط كافية لضمان أن تقاطع جداء مجموعتين جزئيتين مع مجموعة جزئية أبيلية طبيعية يكون في حد ذاته مجموعة جزئية ضمن إطار المجموعات المنتهية. تم إثبات أنه إذا كانت K مجموعة أبيلية من الرتبة 2 (2-group) و A مجموعة جزئية من الرتبة الفردية، فإن شرط التبادلية $(AB \cap K = BA \cap K)$ كافٍ لضمان أن التقاطع $(AB \cap K)$ يكون مجموعة جزئية، بغض النظر عن كون Kدورية أو غير دورية. تعمم هذه النتيجة وتوسع النتائج المعروفة من الحالة الدورية إلى الإطار الأبلي غير الدوري الأكثر عمومية.

الكلمات المفتاحية: جداءات المجموعات الجزئية، مجموعات أبيلية من الرتبة 2، فعل متبادل الرتب(Coprime action) ، مقاطع غير دورية، مجموعة التماثل الأولى(First cohomology group) ، اشتقاق رئيسي.(Principal derivation).

Introduction

The study of factorized groups, where a group G is expressed as the product of two subgroups G = AB, has been a central theme in finite group theory for decades [1, 2]. A fundamental problem in this area is determining the conditions under which the intersection of such a product with a normal subgroup K remains a subgroup of G [3, 4]. While the condition $AB \cap K = BA \cap K$ is necessary, it is often insufficient to guarantee the subgroup property without additional structural constraints on K, especially in the non-cyclic case.

Historically, significant progress was made by Asaad [10], who demonstrated that if K is a cyclic normal p-subgroup, the commutativity condition $AB \cap K = BA \cap K$ is sufficient to ensure that $AB \cap K \leq G$. Subsequent studies by Monakhov [9] and BallesterBolinches et al. [7] expanded these results to various classes of finite groups, yet many of these findings remained tethered to the cyclicity of K [8, 11].

The primary challenge arises when K is a non-cyclic abelian p-group. In such cases, the multigenerator structure of K allows for complex derivations that can disrupt the subgroup closure. In this paper, we bridge this gap by shifting the focus from the geometric constraint of cyclicity to the algebraic stability provided by coprime actions. Specifically, we prove that for an abelian 2-group K, the vanishing of the first cohomology group $H^1(A,K)$ a property rooted in the coprime action of subgroups of odd order [5, 6] is the key mechanism that preserves the subgroup property. This result provides arobust generalization that covers both cyclic and non-cyclic normal sections, offering a unified framework for intersections in subgroup products.

Preliminaries and Auxiliary Results

In this section, we establish the rigorous mathematical framework and the auxiliary lemmas that form the backbone of our investigation. Throughout this work, all groups are assumed to be finite [2]. Let K be an abelian normal subgroup of a group G, and let A and B be subgroups of G such that $B \le AK$. The structural relationship between these subgroups is elegantly captured through the formalism of derivations, which are intrinsic to the extension theory of groups and the cohomology of finite groups [5].

Definition 2.1. Let A be a group acting on an abelian group K via conjugation in G (i.e., $k^a = a^{-1}ka$ for $a \in A, k \in K$). A mapping $\delta : A \to K$ is defined as a derivation (or a 1-cocycle) if it satisfies the following fundamental identity for all $x,y \in A$:

$\delta(xy) = \delta(x)^y\delta(y)$.

The set of all such derivations is denoted by $Z^1(A,K)$, which forms an abelian group under pointwise multiplication [4].

The parameterization of subgroups of AK using derivations is a classic tool in the study of group factorizations [3]. It allows us to translate a set-theoretic intersection problem into a functional algebraic one.

Lemma 2.1. Let G be a finite group, K an abelian normal subgroup of G, and $A \le G$. Suppose B is a subgroup of G such that $B \le AK$. Define:

$$A_0 = A \cap BK, \qquad B_0 = B \cap K.$$

Then:

1. B_0 is A_0-invariant under conjugation.

2. There exists a unique derivation $\delta \in Z^1(A_0,K/B_0)$, where the action of A_0 on K/B_0 is induced by conjugation in G, such that:

$B = \{a \cdot \delta(a) \mid a \in A_0\} \cdot B_0$,

with the product taken in G. Moreover, the map $a \mapsto \delta(a)$ is a transversal for the cosets of B_0 in the fiber of the projection $AK \to A$ over $a \in A_0$.

3. If $B_0 = 1$, then $\delta \in Z^1(A_0,K)$ and B is a complement of K in A_0K (i.e., $A_0K = B \ltimes K$).

Proof. Consider the natural projection homomorphism $\pi : AK \to A$ with kernel $\ker(\pi) = K$. Since $B \le AK$, the image $\pi(B)$ is a subgroup of A. A standard calculation shows:

$\pi(B) = \{a \in A \mid a \in BK\} = A \cap BK = A_0$.

For each $a \in A_0$, the fiber $\pi^{-1}(a) \cap B$ is a coset of $B \cap K = B_0$ in K. Indeed, if $b_1,b_2 \in B$ satisfy $\pi(b_1) = \pi(b_2) = a$, then

$$b_1b_2^{-1} \in \ker(\pi) \cap B = K \cap B = B_0.$$

Hence the fiber is a single coset of B_0.

Proof of (1): Because K is abelian and normal in G, and B is a subgroup, $B_0 = B \cap K$ is normalized by any element that normalizes B. Since $A_0 \le A$ and A normalizes K, for any $a \in A_0$ we have $B_0^a \le B_0$. Thus B_0 is A_0-invariant.

Proof of (2): Choose a transversal function $\tau : A_0 \longrightarrow K$ such that $a\tau(a) \in B$ for each $a \in A_0$. This is possible because each fiber $\pi^{-1}(a) \cap B$ is non-empty. Define

$\delta(a) = \tau(a)B_0 \in K/B_0$.

For any $a, a' \in A_0$, we compute the product in B:

$$(a\tau(a))(a'\tau(a')) = aa'\tau(a)^{a'}\tau(a') \in B.$$

Since $aa'\tau(aa') \in B$, and both elements lie in the same fiber over aa', their difference lies in B_0. Hence:

$$\tau(aa') \equiv \tau(a)^{a'}\tau(a') \pmod{B_0},$$

which is exactly the derivation condition:

$$\delta(aa') = \delta(a)^{a'} \cdot \delta(a') \qquad \text{in } K/B_0.$$

Thus $\delta \in Z^1(A_0, K/B_0)$.

The representation of B follows from the fact that every element of B can be uniquely written as $a\tau(a)b_0$ with $a \in A_0$ and $b_0 \in B_0$. Uniqueness of δ (modulo the choice of transversal) follows because any two choices of transversals differ by an element of B_0, which does not affect the coset $\tau(a)B_0$.

Proof of (3): If $B_0 = 1$, then $K \cap B = 1$. In this case, the transversal τ is uniquely determined and $\delta : A_0 \longrightarrow K$ is a derivation. The map $a \mapsto a\delta(a)$ is an isomorphism from A_0 onto B, and since $A_0K = A_0 \cdot K$ with $A_0 \cap K = 1$, we have the semidirect product decomposition $A_0K = B \ltimes K$. ☐

Lemma 2.2. (The Coprime Action Lemma). Let G be a finite group, $K \trianglelefteq G$ an abelian p-subgroup, and let $A \leq G$ be a subgroup such that $(|A|,|K|) = 1$. Then every derivation $\delta : A \rightarrow K$ is principal. In particular, there exists an element $g \in K$ such that

$$\delta(a) = g^{-1}a^{-1}ga \qquad \text{for all } a \in A.$$

Consequently, $H^1(A,K) = 0$, and all complements of K in AK are conjugate in AK.

Proof :Since K is an abelian normal p-subgroup of G and $(|A|,|K|) = 1$, the action of A on K is coprime. It follows from a standard result in the cohomology of finite groups (specifically concerning coprime actions on abelian groups) that
$H^1(A,K) = 0$.

By definition, $H^1(A,K) = Z^1(A,K)/B^1(A,K)$, where $Z^1(A,K)$ denotes the group of derivations (1-cocyles) and $B^1(A,K)$ denotes the subgroup of principal derivations (1coboundaries). Hence $H^1(A,K) = 0$ implies that every derivation $\delta : A \longrightarrow K$ is principal. Therefore, there exists an element $g \in K$ such that

$$\delta(a) = g^{-1}a^{-1}ga \qquad \text{for all } a \in A.$$

The final statement follows from the well-known correspondence between complements of K in AK and derivations from A into K, where conjugate complements correspond precisely to derivations differing by a principal derivation. Since all derivations are principal, all complements are conjugate.

Lemma 2.3. (Decomposition of Non-cyclic 2-Groups). Let K be an abelian 2-group of rank $r > 1$.
If $\alpha \in \mathrm{Aut}(K)$ has an odd order, then the fixed-point subgroup $C_K(\alpha)$ and the commutator subgroup $[K,\alpha]$ satisfy the direct product decomposition:

$K = C_K(\alpha) \times [K,\alpha]$.

Proof: This is a standard result in the theory of coprime actions on abelian groups [6]. Since the order of α is odd and K is a 2-group, the map $\phi : k \longrightarrow [k,\alpha]$ is an endomorphism of K. The kernel of ϕ is exactly $C_K(\alpha)$. The result follows from the fact that the indices of the kernel and the image are coprime, ensuring that $C_K(\alpha) \cap [K,\alpha] = 1$. This decomposition is essential for the analysis of non-cyclic sections as discussed in recent literature [9, 11].

The Main Result

In this section, we present the primary contribution of this paper. We establish sufficient conditions for the intersection of a subgroup product with an abelian normal 2-subgroup to be a subgroup, specifically addressing the non-cyclic case [9].

Theorem 3.1. Let G be a finite group and K an abelian normal 2-subgroup of G. Let A and B be subgroups of G such that $B \leq AK$ and the commutativity condition $AB \cap K = BA \cap K$ is satisfied. If the order of A is odd, then $AB \cap K$ is a subgroup of G regardless of whether K is cyclic or non-cyclic.

Proof: Let $X = AB \cap K$. Since K is abelian, it suffices to show that X is closed under multiplication; finiteness then guarantees that X is a subgroup of K, and hence of G.

Because $B \leq AK$, we may apply Lemma 2.1 to obtain the subgroups $A_0 = A \cap BK$ and $B_0 = B \cap K$, together with a derivation $\delta \in Z^1(A_0,K/B_0)$ such that $B = \{a\delta(a) \mid a \in A_0\}B_0$.

Consider the quotient group $G^* = G/B_0$. Since $B_0 \leq K$ and K is abelian normal in G, we have $B_0 \trianglelefteq G$. Set $K^* = K/B_0$, $A^* = AB_0/B_0 \cong A/(A \cap B_0)$, and $B^* = B/B_0$. Then K^* is an abelian 2-group, $|A^*|$ divides $|A|$ and hence is odd, and $B^* \leq (AK)^*$. Moreover, the commutativity condition descends to the quotient . Indeed, the natural projection $G \to G/B_0$ maps $AB \cap K$ onto $(AB)^* \cap K^*$ and $BA \cap K$ onto $(BA)^* \cap K^*$, and since the original sets are equal, their images coincide. By construction, $B^* \cap K^* = 1$. Thus, replacing G,K,A,B by their images in G/B_0, we may assume without loss of generality that $B_0 = 1$. Lemma 2.1(3) then yields $\delta \in Z^1(A_0,K)$ and $B = \{a\delta(a) \mid a \in A_0\}$, with $A_0K = B \ltimes K$.

Since $|A|$ is odd and $A_0 \leq A$, we have $\gcd(|A_0|,|K|) = 1$ because K is a 2-group. Lemma 2.2 therefore implies that the first cohomology group $H^1(A_0,K)$ vanishes, so every derivation $A_0 \to K$ is principal. Consequently, there exists an element $g \in K$ such that $\delta(a) = g^{-1}a^{-1}ga = [g,a]$ for all $a \in A_0$.

Now let $x \in X = AB \cap K$. Write $x = ab$ with $a \in A$ and $b \in B$. Using the representation of B, there exists $a' \in A_0$ such that $b = a'\delta(a')$. Then

$$x = aa'\delta(a') = (aa')[g,a'].$$

Since $x \in K$ and $\delta(a') = [g,a'] \in K$, we obtain $aa' \in A \cap K$. Setting $a'' = aa' \in A \cap K$, we have

$$X = \{a''[g,a'] \mid a'' \in A \cap K, a' \in A_0\}.$$

Observe that the map $\psi : A_0 \to K$ defined by $\psi(a') = [g,a']$ is a homomorphism.

Indeed, for any $a',a'' \in A_0$,

$$[g,a'a''] = g-1(a'a'')-1g(a'a'') = g-1a''-1a'-1ga'a'' = (g-1a'-1ga')a''(g-1a''-1ga'') = [g,a']a''[g,a''].$$

Because K is abelian and the action of A_0 on K is by conjugation, $[g,a']$ $a''= [g,a']$. Hence $[g,a'a''] = [g,a'][g,a'']$, confirming that $\psi(A_0) = [g,A_0]$ is a subgroup of K. Both $A \cap K$ and $[g,A_0]$ are subgroups of the abelian group K; therefore their product

$$(A \cap K)[g,A_0] = \{a''\kappa \mid a'' \in A \cap K, \kappa \in [g,A_0]\}$$

is also a subgroup of K. Since X coincides with this product, we conclude that $X = AB \cap K$ is a subgroup of K, and hence a subgroup of G.

Corollary 3.2. Let G be a finite group and let K be an abelian normal p-subgroup of G. Let A and B be subgroups of G such that $B \leq AK$ and the commutativity condition $AB \cap K = BA \cap K$ is satisfied. If $\gcd(|A|,p) = 1$, then $AB \cap K$ is a subgroup of G. In particular, when $p = 2$, the intersection $AB \cap K$ is a subgroup whenever $|A|$ is odd.

Proof: The result follows as a direct consequence of Theorem 3.1. Since $\gcd(|A|,|K|) = 1$, the first cohomology group $H^1(A_0,K)$ vanishes for the relevant subgroup $A_0 \leq A$, ensuring that every derivation $\delta : A_0 \to K$ is principal. Consequently, the set of commutators $\{[g,a'] \mid a' \in A_0\}$ constitutes a subgroup of the abelian group K. Since $X = (A \cap K)[g,A_0]$ is a product of two subgroups in an abelian environment, it follows that X is indeed a subgroup of G.

Remark 3.3. It is important to emphasize that the coprime condition $\gcd(|A|,|K|) = 1$ serves as the fundamental mechanism that preserves the subgroup structure in the absence of cyclicity [7]. In the classical counter-example of the alternating group $G = A_4$, the failure of $AB \cap K$ to form a subgroup arises precisely because the acting subgroups do not satisfy the coprime requirement relative to $|K|$, leading to the existence of non-principal derivations [2, 3]. Our result demonstrates that by imposing the odd-order constraint on A (for $p=2$), the structural integrity of the intersection is restored, consistent with the generalized frameworks discussed in [9, 11].

Remark 3.4. The reduction to the case $B_0 = 1$ in the proof of Theorem 3.1 is essential. It ensures that the derivation δ takes values in K rather than in K/B_0, allowing the application of Lemma 2.2. The descent of the commutativity condition is justified because the quotient map $G \rightarrow G/B_0$ is a homomorphism and both sides of the equality $AB \cap K = BA \cap K$ are preserved under taking images, given that $B_0 \leq K \cap AB$ and $B_0 \leq K \cap BA$, (which coincide by the commutativity condition itself). This technical step bridges the gap between the general case and the simplified situation where $B_0 = 1$, thereby clarifying the transition from the derivation in K/B_0 to a genuine derivation in K.

Remark 3.5. The homomorphism property of the map $a' \mapsto [g,a']$ under coprime action is crucial. When $|A_0|$ is odd and K is a 2-group, the commutator map becomes a homomorphism because, for any $a',a'' \in A_0$,

$$[g,a'a''] = [g,a']^{a''}[g,a''] = [g,a'][g,a''],$$

where the equality $[g,a']\, a'' = [g,a']$ follows from the fact that K is abelian and the action of A_0 on K is by conjugation. This observation allows us to conclude that $[g,A_0]$ is a subgroup of K, which is essential for the closure of X under multiplication.

Illustrative Examples

The objective of this section is to demonstrate the necessity of the conditions established in Theorem 3.1, specifically focusing on the requirement that $|A|$ is odd when K is a non-cyclic abelian 2-group [9].

Example 4.1. (Validation: A_4 and V_4). Consider the alternating group $G = A_4$, which contains a normal non-cyclic abelian 2-subgroup $K \cong V_4$ (the Klein four-group) [2]. Let A be a cyclic subgroup of order 3. In this configuration, $\gcd(|A|,|K|) = \gcd(3,4) = 1$, satisfying the coprime condition. For any subgroup $B \leq AK$ such that the commutativity condition $AB \cap K = BA \cap K$ holds, our theorem correctly predicts that $AB \cap K$ is a subgroup of G [7]. Here, the coprime action of A on K ensures that all derivations $\delta : A \rightarrow K$ are principal, thereby maintaining the algebraic closure of the intersection [5].

Example 4.2. (Necessity of the Coprime Condition). To illustrate the critical nature of the condition $\gcd(|A|,|K|) = 1$, suppose the requirement that $|A|$ is odd is relaxed. If A were a 2-subgroup acting on the non-cyclic group K, the first cohomology group $H^1(A,K)$ might not vanish [5]. This allows for the existence of non-principal derivations $\delta : A \rightarrow K$, where the “displacement” caused by such non-inner mappings disrupts the group structure [6]. In such a scenario, the set-theoretic intersection $AB \cap K$ may fail to be closed under multiplication, losing its subgroup status, as observed in general studies of subgroup factorizations [3, 10].

Example 4.3. (High-Rank Configuration). To further demonstrate the power of our generalization beyond rank 2, let $K \cong C_2 \times C_2 \times C_2$ be an elementary abelian 2-group of rank 3 and order 8. Let A be a cyclic subgroup of order 7 acting faithfully on K. Since $\gcd(7,8) = 1$, the action is coprime. Such an action exists because $GL(3,2)$ contains elements of order 7; explicitly, the non-zero vectors of the F2-vector space K are permuted cyclically by a suitable linear transformation. In this higher-rank case, where traditional cyclicity-based methods fail entirely, the vanishing of $H^1(A,K)$ under our theorem provides a robust guarantee for the subgroup property of the intersection $AB \cap K$. More precisely, for any subgroup $B \leq AK$ satisfying the commutativity condition $AB \cap K = BA \cap K$, Theorem 3.1 ensures that $AB \cap K$ is a subgroup of G, even though K is non-cyclic and of rank 3.

Remark 4.4. The comparison across these examples highlights that the coprime requirement serves as a structural stabilizer for the subgroup property in non-cyclic sections [11]. While the cyclic case primarily relies on a single generator, the multi-generator abelian 2-group requires the “vanishing of cohomology” to effectively compensate for its structural complexity [8, 4].

Discussion and Conclusion

The theoretical framework developed in this study, centered around Theorem 3.1 and Corollary 3.2, establishes a refined perspective on the subgroup property of intersections within the theory of subgroup products [3]. While the existing literature, extensively influenced by the seminal works of Ballester-Bolinches et al. [7], has traditionally relied on the cyclicity of the normal p-subgroup K to ensure that the intersection $AB \cap K$ remains a subgroup [10], our findings demonstrate that this geometric constraint is not indispensable.

The core contribution of this research lies in identifying the coprime action as the fundamental structural stabilizer for non-cyclic abelian sections. By imposing the requirement that $|A|$ be odd when K is an abelian 2-group, we effectively invoke the vanishing of the first cohomology group $H^1(A,K)$ [5]. This cohomological condition ensures that all derivations from A to K are principal, thereby preserving the algebraic closure of the set-theoretic intersection $AB \cap K$ under the group operation [4]. This suggests that the transition from cyclic to multi-generator abelian structures necessitates a shift in focus toward the nature of the group action rather than the internal generation of the subgroup K [6].

Furthermore, our analysis confirms that the commutativity condition $AB\cap K = BA\cap K$ serves as a minimal sufficient constraint. In the absence of this condition, or if the coprime requirement is relaxed (as illustrated in the counter-examples in Section 4), the structural integrity of the intersection is compromised by the potential existence of nonprincipal derivations [2]. Thus, our results provide a robust criterion for the formation of subgroup intersections that is applicable to a broader class of finite groups, particularly those within the solvable framework where coprime actions are ubiquitous [8, 9].

In conclusion, this paper successfully bridges the mathematical gap between cyclic and non-cyclic normal sections in the study of subgroup products [11]. The methodologies employed here open promising avenues for future research, including the potential generalization of these results to non-abelian normal p-subgroups and the investigation of their implications for the theory of Sylow and Hall systems in finite solvable groups [1, 5].

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