African Journal of Advanced Pure and Applied Sciences (AJAPAS)
Online ISSN: 2957-644X
Volume 5, Issue 2, 2026
Page No: 445-454
Website: https://aaasjournals.com/index.php/ajapas/index

ISI 2025: 1.126 **SJIFactor 2024: 6.752** **معامل التأثير العربي: 1.62**

Spectra of Upper Triangular Double-Band Matrices as Operators on the Sequence Space bv_0 with an Application to Atmospheric Physics

Suad H. Abu-Janah *
Department of Mathematics, Faculty of Science, Elmergib University, Msllata, Libya

البنية الدقيقة لطيف المصفوفات المثلثية العلوية ثنائية الأقطار كمؤثرات على فضاء المتتاليات bv_0 مع تطبيق في فيزياء الغلاف الجوي

سعاد هويدي أبو جناح *
قسم الرياضيات، كلية العلوم، جامعة المرقب، مسلاته، ليبيا

*Corresponding author: shabujanah@elmergib.edu.ly

Received: April 17, 2026	Accepted: June 24, 2026	Published: June 29, 2026

Abstract
This study investigates the spectrum and fine spectrum of the generalized difference operator $\triangle^{ab}$ acting on the sequence space bv_0. Although previous research has explored similar operators, the specific spectral properties of $\triangle^{ab}$ on bv_0 have not been fully established. This work extends the existing literature [5] by providing a detailed characterization of these spectral sets and offering a comparative analysis between the new results obtained for the upper triangular operator $\triangle^{ab}$ and previous findings regarding the lower triangular operator $\triangle_{ab}$ on the same space bv_0.This research connects these theoretical results to a practical challenge in atmospheric physics, specifically the vertical analysis of ozone distribution and the dynamics of the Ozone Hole. By leveraging the spectral properties of the generalized difference operator $\triangle^{ab}$ within the space bv_0, we construct a mathematical framework capable of diagnosing density fluctuation and evaluating system stability. This application provides a practical context for our results, showing that the proposed operator is not only mathematically sound but also useful for describing complex environmental dynamics

Keywords: Spectrum, infinite matrices, sequence spaces.

الملخص
تتقصى هذه الدراسة الطيف والطيف الدقيق لمؤثر الفرق المعمم $\triangle^{ab}$ على فضاء المتتاليات bv_0. على الرغم من أن الأبحاث السابقة قد استكشفت مؤثرات مماثلة، إلا أن الخصائص الطيفية المحددة للمؤثر $\triangle^{ab}$ على الفضاء bv_0 لم تُدرس سابقاَ. هذا العمل يعد امتداداَ لأعمال سابقة وذلك من خلال تقديم تحليل مقارن بين النتائج الجديدة التي تم الحصول عليها للمؤثر $\triangle^{ab}$ والنتائج السابقة المتعلقة بطيف مؤثر الفرق المعمم $\triangle_{ab}$ على نفس الفضاء bv_0 في العمل [5]. يربط هذا البحث هذه النتائج النظرية بتحدٍ عملي في فيزياء الغلاف الجوي، وتحديدًا التحليل الرأسي لتوزيع الأوزون وديناميكيات ثقب الأوزون. من خلال الاستفادة من الخصائص الطيفية لمؤثر الفرق المعمم $\triangle^{ab}$ ضمن الفضاء bv_0 نقوم ببناء إطار رياضي قادر على تشخيص تقلبات الكثافة وتقييم استقرار النظام. يوفر هذا التطبيق سياقاً عملياً لنتائجنا، مما يدل على أن العامل المقترح ليس سليمًا رياضياً فحسب، بل إنه مفيد أيضاً لوصف الديناميكيات البيئية المعقدة.

الكلمات المفتاحية: الطيف، المصفوفات اللانهائية، فضاءات المتتابعات.

1. Introduction and basic concepts

The study of the spectra of linear operators is of fundamental importance in functional analysis, as it generalizes the concept of eigenvalues to infinite-dimensional matrices. Spectral values play a crucial role in diverse fields, including linear algebra, differential equations, quantum mechanics and geometry. Determining spectral values is essential not only for theoretical development but also for practical applications, such as analyzing financial data, signal processing, image reconstruction, and understanding resonance dynamics in machine learning.

The primary objective of this paper is to conduct a comprehensive spectral investigation of the upper generalized difference operator $\triangle^{ab}$ on the sequence space bv_0. Analytically, the operator $\triangle^{ab}: bv_0 \to bv_0$ is defined by

$$\triangle^{ab} x := (a_k x_k + b_k x_{k+1})_{k=0}^{\infty}, \ x = (x_k)_{k=0}^{\infty} \in bv_0,$$

and can be represented by the upper triangular double-band matrix

$$\triangle^{ab} = \begin{bmatrix} a_0 & b_0 & 0 & 0 & \cdots \\ 0 & a_1 & b_1 & 0 & \cdots \\ 0 & 0 & a_2 & b_2 & \cdots \\ 0 & 0 & 0 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \qquad \textbf{(1.1)}$$

where (a_k) and (b_k) are convergent sequences of nonzero real numbers such that

$\lim_{k\to\infty} a_k = a$ and $\lim_{k\to\infty} b_k = b \neq 0$.

To guarantee the boundedness of the operator $\triangle^{ab}$ within bv_0, we necessitate the following variation condition.

$$\sup_N \sum_{i=1}^{N} |(a_i - a_{i-1}) + (b_i - b_{i-1})| < \infty \qquad \textbf{(1.2)}$$

This work complements and extends previous studies, specifically the determination of the fine spectrum of upper triangular double-band matrices over the sequence spaces $\ell_p, (1 < p < \infty)$ and ℓ_1 by A. Karaisa [7,8], the investigation of generalized upper double-band matrices $\triangle^{uv}$ over the sequence space ℓ_1 by J. Fathi and R. Lashkaripour [6], and the study of the fine structure of spectra of the operator $\triangle^{ab}$ over the sequence space ℓ_p, where $1 \leq p < \infty$ by S. El-Shabrawy and S. Abu-Janah[4], (see also [3,9,16]).

The definitions of some sequence spaces are needed in this work.

Definition 1: The sequence spaces bs of bounded series and bv_0 of all sequences of bounded variation are defined as follows:

$$bs = \{x = (x_k)_{k=0}^{\infty} \in w: \ x_k \in \mathbb{C}, \ \sup_{n\in\mathbb{N}} |\textstyle\sum_{k=0}^{n} x_k| < \infty\},$$

$$bv_0 = \{x = (x_k)_{k=0}^{\infty} \in w, \ x_k \in \mathbb{C}, \text{such that } \lim_{k\to\infty} x_k = 0, \ \textstyle\sum_{k=0}^{\infty} |x_{k+1} - x_k| < \infty\}.$$

equipped with the norms:

$$\|x\|_{bs} = \sup_{n\in\mathbb{N}} |\textstyle\sum_{k=0}^{n} x_k|, \qquad \|x\|_{bv_0} = \textstyle\sum_{k=0}^{\infty} |x_{k+1} - x_k|.$$

where w is the set of all complex sequences and $\mathbb{C}$ denotes the complex field [11,17].

If T is an operator on a space X into itself, the spectrum $\sigma(T, X)$ is analyzed into three disjoint sets: the point spectrum $\sigma_p(T, X)$, the residual spectrum $\sigma_r(T, X)$, and the continuous spectrum $\sigma_c(T, X)$, see more in [13].

An advantage of this classification is the division of the spectrum into disjoint sets.

$$\sigma(T, X) = \sigma_p(T, X) \cup \sigma_c(T, X) \cup \sigma_r(T, X).$$

Another classification of the spectrum is also considered, see Taylor and Halberg [14,15]. In this framework, the spectrum is subdivided into finer classes, we specifically focus on the class $II_2(T, X)$.

Three more subdivisions of the spectrum can be defined: the approximate point spectrum $\sigma_{ap}(T,X)$, the defect spectrum $\sigma_{\delta}(T,X)$ and the compression spectrum $\sigma_{co}(T,X)$, see [2].

The following lemmas are established to support the subsequent proofs regarding operator boundedness and spectral properties of $\triangle^{ab}$ on bv_0.

Lemma 1: If T is a bounded linear operator on a Banach space X into itself, then

$$\sigma_r(T,X) = \sigma_p(T^*,X^*)\backslash\sigma_p(T,X).$$

Lemma 2: [15, Page 59]. If T is a bounded linear operator on a normed space X into a normed space Y, then T has a dense range in Y if and only if $(T^*)^{-1}$ exists.

Lemma 3: [15, Page 60]. T has a bounded inverse if and only if T^* is onto.

Lemma 4: [12, Formula 111] An infinite matrix $A = (a_{nk})$, $n \geq 0, k \geq 0$ defines an operator $T \in B(bv_0)$ if and only

(i) $\lim\limits_{k\to\infty} a_{nk} = 0$, for all $k \in N$.

(ii) $\|A\|_{bv_0} := \sup\limits_{N} \sum_{n=0}^{\infty}\left|\sum_{k=0}^{N}(a_{nk} - a_{n-1,k})\right| < \infty$.

where ℕ is the set of nonnegative integers.

2. Main results

The boundedness of the operator $\triangle^{ab}$ on the sequence space bv₀ can be proved as follows.

Theorem 1: The operator $\triangle^{ab}: bv_0 \to bv_0$ is a bounded linear operator.

Proof. We can arrive at the conclusion by applying Lemma 4.

The operator $\triangle^{ab}$ can be represented by the matrix in (1.1). Obviously, it is linear and $\lim\limits_{n\to\infty} c_{nk} = 0$, for any $k \in \mathbb{N}$. Additionally, let

$$A_N = \sum_{n=0}^{\infty}\left|\sum_{k=0}^{N}(c_{nk} - c_{n-1,k})\right|, n \in \mathbb{N}.$$

For $N = 0$, we have

$$\begin{aligned} A_0 &= \sum_{n=0}^{\infty}\left|\sum_{k=0}^{0}(c_{nk} - c_{n-1,k})\right| \\ &= |a_0 - 0| + |0 - a_0| + 0 + \cdots \\ &= 2|a_0|. \end{aligned}$$

For $N = 1$, we have

$$\begin{aligned} A_1 &= \sum_{n=0}^{\infty}\left|\sum_{k=0}^{1}(c_{nk} - c_{n-1,k})\right| \\ &= |a_0 + b_0| + |(a_1 - a_0) - b_0| + |a_1|. \end{aligned}$$

For $N = 2$, we have

$$\begin{aligned} A_2 &= \sum_{n=0}^{\infty}\left|\sum_{k=0}^{2}(c_{nk} - c_{n-1,k})\right| \\ &= |a_0 + b_0| + |(a_1 - a_0) + (b_1 - b_0)| + |(a_2 - a_1) - b_1| + |a_2|. \end{aligned}$$

For $N = 3$, we have

$$\begin{aligned} A_3 &= \sum_{n=0}^{\infty}\left|\sum_{k=0}^{3}(c_{nk} - c_{n-1,k})\right| \\ &= |a_0 + b_0| + |(a_1 - a_0) + (b_1 - b_0)| + |(a_2 - a_1) + (b_2 - b_1)| + |(a_3 - a_2) - b_2| \\ &\quad + |a_3|. \end{aligned}$$

Then, by induction, we have

$$\begin{aligned} A_N &= |a_0 + b_0| + |(a_1 - a_0) + (b_1 - b_0)| + |(a_2 - a_1) + (b_2 - b_1)| \\ &\quad + \cdots + |(a_{N-1} - a_{N-2}) + (b_{N-1} - b_{N-2})| + |(a_N - a_{N-1}) - b_{N-1}| + |a_N| \\ &\le |a_0 + b_0| + \sum_{i=0}^{N} |(a_i - a_{i-1}) + (b_i - b_{i-1})| + |b_N| + |a_N|, \end{aligned}$$

for all $N \geq 1$. So

$$\sup_N A_N \leq \sup_N |a_0 + b_0| + \sup_N \sum_{i=0}^{N} |(a_i - a_{i-1}) + (b_i - b_{i-1})| + \sup_N |b_N| + \sup_N |a_N|,$$

The convergence of the sequence (a_k) and (b_k), and the condition (1.2) imply that $\sup_N A_N < \infty$. ■

The point spectrum of the operator $\triangle^{ab}$ can be investigated as follows.

Theorem 2:

$$\sigma_p(\triangle^{ab}, bv_0) = \{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \cup E \cup A,$$

where

$$A = \{\lambda \in \mathbb{C} : |\lambda - a| = |b|, \sum_{k=0}^{\infty} \left|\frac{\lambda - a_k}{b_k} - 1\right| \left|\prod_{i=0}^{k-1} \frac{\lambda - a_i}{b_i}\right| < \infty, \qquad \textbf{(2.1)}$$

and E is given as

$$E = \{a_k : k \in \mathbb{N}, |a_k - a| > |b|\}. \qquad \textbf{(2.2)}$$

Proof: Suppose that $\triangle^{ab} x = \lambda x$ for $x = (x_0, x_1, x_2, \dots) \neq \theta$ in bv_0. Then, by solving the system of equations

$$\begin{aligned} a_0 x_0 + b_0 x_1 &= \lambda x_0 \\ a_1 x_1 + b_1 x_2 &= \lambda x_1 \\ a_2 x_2 + b_2 x_3 &= \lambda x_2 \\ &\vdots \\ a_k x_k + b_k x_{k+1} &= \lambda x_k \\ &\vdots \end{aligned}$$

we have

$$x_{k+1} = \frac{\lambda - a_k}{b_k} x_k, k \in \mathbb{N},$$

and $\lambda = a_k$ is the eigenvalue corresponding to eigenvector $x = (x_0, x_1, x_2, \dots, x_k, 0, 0, \dots)$ with $x_0 \neq 0$. Then

$$\{a_k : k \in \mathbb{N}\} \subseteq \sigma_p(\triangle^{ab}, bv_0).$$

On the other hand, if $\lambda \neq a_k$ for all $k \in \mathbb{N}$, then $x_k \neq 0$ for all $k \in \mathbb{N}$, so, if $\lambda \in \{\lambda \in \mathbb{C} : |\lambda - a| < |b|\}$, then

$$\lim_{k \to \infty} \left|\frac{x_{k+1}}{x_k}\right| = \lim_{k \to \infty} \left|\frac{\lambda - a_k}{b_k}\right| = \left|\frac{\lambda - a}{b}\right| < 1,$$

so $\lim_{k \to \infty} x_k = 0$. Also, if $\lambda \in A$, that is $|\lambda - a| = |b|$ and

$$\sum_{k=0}^{\infty} |x_{k+1} - x_k| = \sum_{k=0}^{\infty} \left|\frac{\lambda - a_k}{b_k} - 1\right| \left|\prod_{i=0}^{k-1} \frac{\lambda - a_i}{b_i}\right| < \infty.$$

Thus $x = (x_k)_{k=0}^{\infty} \in bv_0$. So we can take $x \in bv_0$, $x \neq \theta$ with $\triangle^{ab} x = \lambda x$, that is

$\lambda \in \sigma_p(\triangle^{ab}, bv_0)$.Hence

$$\{\lambda \in \mathbb{C}: |\lambda - a| < |b|\} \cup E \cup A \subseteq \sigma_p(\triangle^{ab}, bv_0).$$

Conversely, let $\lambda \in \sigma_p(\triangle^{ab}, bv_0)$, then, there exists $x = (x_k)_{k=0}^{\infty} \in bv_0$, $x \neq \theta$ such that $\triangle^{ab} x = \lambda x$. Then

$$\lim_{k\to\infty} \left|\frac{x_{k+1}}{x_k}\right| = \lim_{k\to\infty} \left|\frac{\lambda - a_k}{b_k}\right| = \left|\frac{\lambda - a}{b}\right| \leq 1.$$

Consequently, $\lambda \in \{\lambda \in \mathbb{C}: |\lambda - a| < |b|\}$ or $|\lambda - a| = |b|$. In the case where

$|\lambda - a| = |b|$, we must have $\sum_{k=0}^{\infty} |x_{k+1} - x_k| < \infty$, that is

$$\sum_{k=0}^{\infty} \left|\frac{\lambda - a_k}{b_k} - 1\right| \left|\prod_{i=0}^{k-1} \frac{\lambda - a_i}{b_i}\right| < \infty.$$

So $\lambda \in A$.

Also $\lambda \in E$. Then

$\sigma_p(\triangle^{ab}, bv_0) \subseteq \{\lambda \in \mathbb{C}: |\lambda - a| < |b|\} \cup E \cup A$. ■

Theorem 3:

$$\sigma_p(\triangle^{ab*}, bv_0{}^*) = E \cup B,$$

Where E is given as in (2.2) and

$$B = \left\{a_j : j \in \mathbb{N},\ |a_j - a| = |b|, \sup_{n \geq m_{j+1}} \left|\sum_{k=m_{j+1}}^{n} \left(\prod_{i=m_j}^{k} \frac{b_{i-1}}{a_j - a_i}\right)\right| < \infty\right\}, \quad \textbf{(2.3)}$$

Where $m_J \in \mathbb{N}$ for any fixed $j \in \mathbb{N}$ with $a_j \neq a$, such that $a_j - a_k \neq 0$ for all $k > m_j$, (so, if $|a_j - a| = |b| \neq 0$, then there exists $m_j \in \mathbb{N}$ such that $a_j - a_k \neq 0$ for all $k > m_j$).

Proof. Consider $\triangle^{ab*} f = \lambda f$ for $f = (f_0, f_1, f_2, \dots) \neq \theta$ in $bv_0{}^* \cong bs$ [11]. Then by solving the system of equations

$$\begin{aligned}
(a_0 - \lambda) f_0 &= 0 \\
b_0 f_0 + (a_1 - \lambda) f_1 &= 0 \\
b_1 f_1 + (a_2 - \lambda) f_2 &= 0 \\
&\vdots \\
b_k f_k + (a_{k+1} - \lambda) f_{k+1} &= 0. \\
&\vdots
\end{aligned}$$

we obtain

$$f_{k+1} = \frac{b_k}{\lambda - a_{k+1}} f_k, \quad k \geq 0.$$

It is clear that, if $\lambda \notin \{a_k : k \in \mathbb{N}\}$, we have $f_k = 0$ for every $k \in \mathbb{N}$.

So $\lambda \notin \sigma_p(\triangle^{ab*}, bv_0{}^*)$. Also, if $\lambda = a$, this implies that the sequence $f = (f_k)$ is not convergent, that is $\lambda \notin \sigma_p(\triangle^{ab*}, bv_0{}^*)$. Thus

$$\sigma_p(\triangle^{ab*}, bv_0{}^*) \subseteq \{a_k : k \in \mathbb{N}\} \backslash \{a\}.$$

Therefore, if $\lambda \in \sigma_p(\triangle^{ab*}, {bv_0}^*)$, then $\lambda = a_j \neq a$ for some $j \in \mathbb{N}$ and there exists

$f \in {bv_0}^*, f \neq \theta$ such that $\triangle^{ab*} f = a_j f$. So, we have

$$\lim_{k\to\infty}\left|\frac{f_{k+1}}{f_k}\right| = \lim_{k\to\infty}\left|\frac{b_k}{a_j - a_{k+1}}\right| = \left|\frac{b}{a_j - a}\right| \leq 1.$$

This implies that $\lambda = a_j \in E$ or $|a_j - a| = |b|$. In the case when $|a_j - a| = |b|$, we have

$$f_k = \frac{b_{k-1}b_{k-2}\cdots b_{m-1}}{(a_j - a_k)(a_j - a_{k-1})(a_j - a_{k-2})} f_{m-1}$$

$$= f_{m-1}\prod_{i=m_j}^{k}\frac{b_{i-1}}{a_j - a_i}, \qquad k \geq m.$$

So $\sup_{n\geq 0}\left|\sum_{k=0}^{n} f_k\right| < \infty$ if

$$\sup_{n\geq m_{j+1}}\left|\sum_{k=m_{j+1}}^{n}\left(\prod_{i=m_j}^{k}\frac{b_{i-1}}{a_j - a_i}\right)\right| < \infty$$

Thus $\sigma_p(\triangle^{ab*}, {bv_0}^*) \subseteq E \cup B$.

Conversely, let $\lambda \in E \cup B$. If $\lambda \in E$, then there exists $i \in \mathbb{N}$ such that $\lambda = a_i \neq a$ and so we can take $f \neq \theta$ such that $\triangle^{ab*} f = a_i f$ and

$$\lim_{k\to\infty}\left|\frac{f_{k+1}}{f_k}\right| = \left|\frac{b}{a_j - a}\right| < 1,$$

that is $f \in {bv_0}^* \cong bs$. Also, if $\lambda \in B$, then there exists $j \in \mathbb{N}$ such that $\lambda = a_j \neq a$, $|a_j - a| = |b|$ and

$$\sup_{n\geq m_{j+1}}\left|\sum_{k=m_{j+1}}^{n}\left(\prod_{i=m_j}^{k}\frac{b_{i-1}}{a_j - a_i}\right)\right| < \infty, \text{ for some } m \in \mathbb{N}.$$

Then, we can take $f \in {bv_0}^* \cong bs, f \neq \theta$ such that $\triangle^{ab*} f = a_j f$.

Thus $E \cup B \subseteq \sigma_p(\triangle^{ab*}, {bv_0}^*)$. ■

Theorem 4: Without canceling the general situation, we may assume additionally the following condition:

$|a_j - a| \neq |b|$, for all $j \in \mathbb{N}$.

In this case $B = \emptyset$. So, we have

$$\sigma_r(\triangle^{ab}, bv_0) = \emptyset.$$

Proof. The proof obtained by lemma 1 and theorems 2, 3.

Remark: In general situation, without the condition

$$|a_j - a| \neq |b|, \text{ for all } j \in \mathbb{N},$$

so, $B \neq \emptyset$. Then

$$\sigma_r(\triangle^{ab}, bv_0) = B \setminus (\{\lambda \in \mathbb{C}: |\lambda - a| < |b|\} \cup A).$$

Theorem 5:

$$II_2(\triangle^{ab}, bv_0) = \{\lambda \in \mathbb{C}: |\lambda - a| = |b|\} \setminus A,$$

where A is given as in (2.1).

Proof. If $\lambda \in \{\lambda \in \mathbb{C}: |\lambda - a| = |b|\} \backslash A$, then $\lambda \notin \sigma_p(\triangle^{ab}, bv_0)$, so $(\triangle^{ab} - \lambda I)^{-1}$ exists. Since $\sigma_p(\triangle^{ab*}, {bv_0}^*) \subseteq \sigma_p(\triangle^{ab}, bv_0)$, then $(\triangle^{ab*} - \lambda I)^{-1}$ exists, so by Lemma 2, $(\triangle^{ab} - \lambda I)$ has a dense range and then we must show that $(\triangle^{ab*} - \lambda I)$ is onto for $|\lambda - a| > |b|$ because $\lambda \notin \sigma_p(\triangle^{ab}, bv_0)$. Also, if $(\triangle^{ab*} - \lambda I)^{-1} = (c_{nk})$, where (c_{nk}), is represented by the matrix in (1.1), then $\sum_{k=0}^{\infty} |x_k| \leq \sup_k R_k \sum_{k=0}^{\infty} |y_k|$, where

$$R_k = \frac{1}{|a_k - \lambda|} + \frac{|b_{k-1}|}{|a_{k-1} - \lambda|\,|a_k - \lambda|} + \frac{|b_{k-2}|\,|b_{k-1}|}{|a_{k-2} - \lambda|\,|a_{k-1} - \lambda|\,|a_k - \lambda|} + \cdots + \frac{|b_0|\,|b_1| \cdots |b_{k-1}|}{|a_0 - \lambda|\,|a_1 - \lambda| \cdots |a_k - \lambda|}, k \in \mathbb{N}.$$

Since $\lim_{k\to\infty} \left|\frac{b_k}{a_k-\lambda}\right| = |\frac{b}{a-\lambda}| < 1$, $(|\frac{b}{a-\lambda}| \neq 1)$, then, there exists $k_0 \in \mathbb{N}$ and $d < 1$ such that $\left|\frac{b_k}{a_k-\lambda}\right| < d$, for all $k \geq k_0$. Then, for each $k > k_0$, using the similar arguments in the corresponding theorems in[1,7], we get

$$R_k \leq \frac{1}{|a_{k-1}-\lambda|}[1 + d_0 + {d_0}^2 + \cdots + {d_0}^{k-k_0} h_{k_0}],$$

$$h_{k_0} = 1 + \frac{|b_{k_0-1}|}{|a_{k_0-1}-\lambda|} + \frac{|b_{k_0-1}|\,|b_{k_0-2}|}{|a_{k_0-1}-\lambda|\,|a_{k_0-2}-\lambda|} + \cdots + \frac{|b_{k_0-1}|\,|b_{k_0-2}| \cdots |b_0|}{|a_{k_0-1}-\lambda|\,|a_{k_0-2}-\lambda|\,|a_0-\lambda|}.$$

so

$$R_k \leq \frac{h_{k_0}}{|a_k-\lambda|}[1 + d_0 + {d_0}^2 + \cdots + {d_0}^{k-k_0}].$$

Then $R_k \leq \frac{d_1 h_{k_0}}{1-d_0}$, for all $k > max\{k_0, k_1\}$, where $d_1 < \frac{1}{|b|}$, $d_1 \in \mathbb{R}$ and $\frac{1}{|a_k-\lambda|} < d_1$ for all $k \geq k_1$, $k_1 \in \mathbb{N}$. Thus $\sup_k R_k < \infty$, which leads to $\sum_{k=0}^{\infty} |x_k| < \infty$, so

$$|\sum_{k=0}^{n} x_k| \leq \sum_{k=0}^{n} |x_k| < \sum_{k=0}^{\infty} |x_k| < \infty.$$

Thus $\sup_n |\sum_{k=0}^{n} x_k| < \infty$, that is $x = (x_k)_{k=0}^{\infty} \in bs$. Hence $(\triangle^{ab*} - \lambda I)^{-1}$ is onto for $|\lambda - a| > |b|$. Then by Lemma 3, $(\triangle^{ab} - \lambda I)$ has a bounded inverse if $|\lambda - a| > |b|$.

So $\lambda \in II_2(\triangle^{ab}, bv_0)$. Then

$$\{\lambda \in \mathbb{C}: |\lambda - a| = |b|\} \backslash A \subseteq II^{2(\triangle^{ab}, bv_0)}.$$

The second inclusion can be proved analogously. ■

Theorem 6:

$$\sigma_c(\triangle^{ab}, bv_0) = \{\lambda \in \mathbb{C}: |\lambda - a| = |b|\} \backslash A,$$

where A is given as in (2.1).

Proof. The proof may be obtained by the relation

$$\sigma_c(\triangle^{ab}, bv_0) = II_2(\triangle^{ab}, bv_0).$$ ■

Theorem 7:

$$\sigma(\triangle^{ab}, bv_0) = \{\lambda \in \mathbb{C}: |\lambda - a| \leq |b|\} \cup E$$

where, E is given as in (2.2).

Proof. The proof is obtained by theorems 2, 4, 6 and relation

$$\sigma(\triangle^{ab}, bv_0) = \sigma_p(\triangle^{ab}, bv_0) \cup \sigma_r(\triangle^{ab}, bv_0) \cup \sigma_c(\triangle^{ab}, bv_0).$$ ■

Consequently

$III_1(\triangle^{ab}, bv_0) = III_2(\triangle^{ab}, bv_0) = \emptyset$, since $\sigma_r(\triangle^{ab}, bv_0) = \emptyset$,

and $I_2(\triangle^{ab}, bv_0) = \emptyset$, by the closed graph theorem.

Theorem 8:

1. $I_3(\triangle^{ab}, bv_0) \cup II_3(\triangle^{ab}, bv_0) = \sigma_p(\triangle^{ab}, bv_0) \setminus \sigma_p(\triangle^{ab*}, {bv_0}^*)$
$= \{\lambda \in \mathbb{C}: |\lambda - a| < |b|\} \cup (A \setminus B),$
2. $III_3(\triangle^{ab}, bv_0) = \sigma_p(\triangle^{ab*}, {bv_0}^*) = E \cup B,$
3. $.\sigma_{ap}(\triangle^{ab}, bv_0) = \{\lambda \in \mathbb{C}: |\lambda - a| \leq |b|\} \cup E,$
4. $\sigma_{co}(\triangle^{ab}, bv_0) = E \cup B,$
5. $\sigma_{\delta}(\triangle^{ab}, bv_0) = [\{\lambda \in \mathbb{C}: |\lambda - a| \leq |b|\} \cup E] \setminus I_3(\triangle^{ab}, bv_0),$

where, E, A and B are given as in (2.2), (2.1) and (2.3), respectively.

To demonstrate the theoretical findings established in Theorems 2, 4, 6 and 7, we present a numerical case for the operator $\triangle^{ab}$ on the sequence space bv_0.

Example 1. Let $(a_k) = (1,1,1,\dots)$, $(b_k) = (\frac{1}{2}, \frac{1}{2}, \dots)$. Then, $a = 1$, $b = \frac{1}{2}$.

According to the conditions established in our previous theorems, the point spectrum is determined by finding $\lambda \in \mathbb{C}$ such that the equation $(\triangle^{ab} - \lambda I)x = 0$ has a non-zero solution $x \in bv_0$. This leads to the recurrence relation:

$$(1 - \lambda)x_k + \frac{1}{2}x_{k+1} = 0 \Rightarrow \quad x_{k+1} = 2(\lambda - 1)x_k.$$

For $x \in bv_0$, the sequence must satisfy $\lim_{k\to\infty} x_k = 0$ and be of bounded variation. This occurs when $|2(\lambda - 1)| < 1$. Thus,

$$\sigma_p(\triangle^{ab}, bv_0) = \left\{\lambda \in \mathbb{C}: |\lambda - 1| < \frac{1}{2}\right\},$$

the characterization of the residual spectrum is one of the key findings of this study, we observe that the range of $(\triangle^{ab} - \lambda I)$ is dense in bv_0 for all $\lambda \in \sigma(\triangle^{ab}, bv_0) \setminus \sigma_p(\triangle^{ab}, bv_0)$. Consequently, we verify that

$$\sigma_r(\triangle^{ab}, bv_0) = \emptyset,$$

the continuous spectrum consists of the boundary of the disk where the operator is not invertible but has a dense range. So,

$$\sigma_c(\triangle^{ab}, bv_0) = \left\{\lambda \in \mathbb{C}: |\lambda - 1| = \frac{1}{2}\right\},$$

the total spectrum of the operator is the closed disk

$$\sigma(\triangle^{ab}, bv_0) = \left\{\lambda \in \mathbb{C}: |\lambda - 1| \leq \frac{1}{2}\right\}.$$

3. Application of the main Results: Insights into Ozone Distribution

Following the theoretical development established in the previous sections, we apply the spectral results of the operator $\triangle^{ab}$ on bv_0 to a practical application. By representing the ozone density at different altitudes as a sequence $x = (x_k)_{k=0}^{\infty} \in bv_0$, this mathematical structure helps us study ozone distribution, where the condition $\lim_{k\to\infty} x_k = 0$ shows how ozone density drops continuously due to pressure decay. At the same time, the bounded variation property ensures vertical profile stability, meaning that ozone levels do not show sudden changes between adjacent layers. To analyze these interconnected atmospheric layers, the operator $\triangle^{ab}$ limits the density variations by assuming that chemical exchange at index k only influences the immediate succeeding state $k + 1$. The main diagonal coefficients (a_k) represents independent local factors, such as ozone production and destruction rates within the same layer, while the superdiagonal coefficients (b_k) determines the transport and continuous diffusion process. This mathematical structure gives clear physical results through spectral classification. The point spectrum represents resonance states and isolated zones of static equilibrium, where discrete eigenvalues correspond to specific altitudes experiencing either a notable increase in baseline ozone or sudden chemical destruction. Conversely, the continuous spectrum describes continuous, macroscopic dynamical phenomena that extend across different layer boundaries, such as free gas diffusion driven by wind fields, rising thermal currents, and sequential photochemical chains. Lastly, the residual spectrum covers unsteady states and total energy loss within the system due to external disturbances, where the injectivity of the operator ensures this component contains no eigenvalues, representing localized perturbations that lack a dense range, fail to span the entire atmospheric domain, and prevent standard resonance. The physical information and data regarding ozone distribution used in this application are obtained from London [10] and WMO [18].

Example 2. Let $a_k = 1 + e^k$ and $b_k = \frac{1}{2}\left(1 - \frac{1}{k+1}\right)$, $k = 0,1,2, \dots$.

From these sequences, we have $a = \lim_{k\to\infty} a_k = 1$ and $b = \lim_{k\to\infty} b_k = \frac{1}{2}$.

The set E, defined by the condition $|a_k - a| > |b|$, yields $E = \{2\}$ since the condition $e^{-k} > 0.5$ is satisfied only for $k = 0$. Furthermore, the set A is found to be empty ($A = \emptyset$) because the series associated with the coefficients in (2.1) does not converge for any λ on the boundary of the disk for this specific structure. Consequently, the point spectrum is determined as $\sigma_p(\triangle^{ab}, bv_0) = \left\{\lambda \in \mathbb{C}: |\lambda - 1| < \frac{1}{2}\right\} \cup \{2\}$, suggesting resonant states at specific altitudes where chemical depletion is significant. As a direct application of Theorem 4, the residual spectrum is empty ($\sigma_r(\triangle^{ab}, bv_0) = \emptyset$) since $|a_j - a| \neq |b|$ for all $j \in \mathbb{N}$ and $A = \emptyset$, implying that the range of the operator is dense in bv_0.

The continuous spectrum is identified as the boundary $\sigma_c(\triangle^{ab}, bv_0) = \left\{\lambda \in \mathbb{C}: |\lambda - 1| = \frac{1}{2}\right\}$, and the total spectrum is represented by the closed disk $\sigma(\triangle^{ab}, bv_0) = \left\{\lambda \in \mathbb{C}: |\lambda - 1| \leq \frac{1}{2}\right\} \cup \{2\}$.

Conclusion and Recommendations:

In this study, we have determined the spectrum and its fine subdivisions for the double-band operator $\triangle^{ab}$ on the sequence space bv_0. Our results show that the spectral boundaries are defined by the limit points of the involved sequences. These results represent an initial effort to apply pure mathematics in describing atmospheric phenomena, such as the distribution of the ozone layer in the atmosphere, offering a basis for developing more advanced operators in future studies, such as quadruple-band operators. By looking at the limit points, we can try to better understand how ozone concentrations change and how they are layered, which may assist researchers in atmospheric physics during data analysis. Accordingly, it is recommended to further generalize the investigated operator by increasing its diagonal complexity. Such an expansion toward multi-diagonalization can be used to offer a flexible mathematical tool for characterizing non-linear shifts in ozone distribution, while specific structural conditions on its defining sequences can directly enhance the precision of multi-layer atmospheric transport flux simulations.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

References:

[1] A. M. Akhmedov, S. R. El-Shabrawy, On the spectrum of the generalized difference operator $\triangle_{ab}$ over the sequence space c_0, Baku Univ. News J., Phys. Math. Sci. Ser. 4 (2010), 12-21.

[2] J. Appell, E. De Pascale, A. Vignoli, Nonlinear Spectral Theory; de Gruyter Series in Nonlinear Analysis and Applications 10, Walter de Gruyter, Berlin, 2004.

[3] E. Dündar, F. Başar, On the fine spectrum of the upper triangle double band matrix $\triangle^{+}$ on the sequence space c_0, Math. Commun. 18 (2013), 337-348.

[4] S. R. El-Shabrawy, S. H. Abu-Janah, On the fine structure of spectra of upper triangular double-band matrices as operators on □□□spaces, Appl. Math. Inf. Sci. 10 (3) (2016), 1161-1167.

[5] S. R. El-Shabrawy, S. H. Abu-Janah, Spectra of the generalized difference operator on the sequence spaces bv_0 and h, Linear and Multilinear Algebra, accepted manuscript, 2017, 1-18.

[6] J. Fathi, R. Lashkaripour, On the fine spectrum of generalized upper double-band matrices $\triangle^{uv}$ over the sequence space ℓ_1, Math. Vesnik. 65 (1) (2013), 64-73.

[7] A. Karaisa, Fine spectra of upper triangular double-band matrices over the sequence space □□, $(1 < p < \infty)$, Discrete Dyn. Nat. Soc., 2012, Art. ID 381069, 19 pp.

[8] A. Karaisa, Spectrum and fine spectrum of generalized difference operator over the sequence space ℓ_1, Math. Sci. Lett. 3 (2014), 215-221.

[9] V. Karakaya, M. Altun, Fine spectra of upper triangular double-band matrices, J. Comput. Appl. Math. 234 (2010), 1387-1394.

[10] J. London, Radiative Transfer and the Distribution of Atmospheric Ozone. Academic Press, New York, 1980.

[11] I. J. Maddox, Elements of Functional Analysis, Cambridge Univ. Press, London, 1970.

[12] M. Stieglitz, H. Tietz, Matrixtranformationen von folgenräumen eine ergebnisubersicht, Math. Z. 154 (1977), 1-16.

[13] M. H. Stone, Linear Transformations in Hilbert Space and their Applications to Analysis, American Mathematical Society, New York, 1932.

[14] A. E. Taylor, Introduction to Functional Analysis, John Wiley & Sons, Inc., New York, 1958.

[15] A. E. Taylor, C. J. A. Halberg, Jr., General theorems about a bounded linear operator and its conjugate, J. Reine Angew. Math. 198 (1957), 93-111.

[16] B. C. Tripathy, R. Das, Spectrum and fine spectrum of the upper triangular matrix $U(r,s)$ over the sequence space cs, Proyecciones J. Math. 34 (2) (2015), 107-125.

[17] A. Wilansky, Summability Through Functional Analysis, North-Holland Mathematics Studies, vol. 85, Amsterdam, New York, Oxford, 1984.

[18] World Meteorological Organization(WMO), Scientific Assessment of Ozone Depletion:2022, GAW Report No.278, Geneva, 2022.

