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Weak Derivatives and Their Role in the Construction of Sobolev Spaces

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المشتقات الضعيفة ودورها في بناء فضاءات سوبوليف

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Abstract
This paper studies weak derivatives as a generalization of classical derivatives defined in an integral sense. By introducing test functions and L^p spaces. We prove that weak derivatives coincide with classical derivatives for smooth functions and are unique up to sets of measure zero. Through particular examples, we show that weak derivatives can exist even when classical differentiability fails, and we identify how discontinuity can prevent their existence. We then define Sobolev spaces as L^p functions with weak derivatives in L^p, and prove that they are Banach spaces. Finally, we establish that weak derivatives keep the main properties of classical derivatives, such as linearity and Leibniz's formula, under mild assumption.

Keywords: Smooth functions; Test Functions; support of a function; weak derivative; Sobolev spaces; Absolute continuity; Lipschitz functions.

الملخص:
تدرس هذه الورقة المشتقات الضعيفة كتعميم للمشتقات الكلاسيكية، عن طريق دوال الاختبار وفضاءات L^p. نثبت أن المشتقات الضعيفة تطابق المشتقات الكلاسيكية للدوال الناعمة، ونبرهن وحدانيتها عدا على مجموعات ذات قياس صفر. كما نوضح عبر أمثلة علمية أن المشتقات الضعيفة تكون موجودة عندما يفشل الاشتقاق الكلاسيكي، ونبين ان عدم الاستمرارية قد تمنع وجود المشتقة الضعيفة. بعد ذلك نعرف فضاءات سوبوليف بأنها فضاءات دوال L^p التي تنتمي مشتقاتها الضعيفة أيضا الى L^p، ونثبت أنها فضاءات باناخ. وأخيرا نتوصل الى أن المشتقات الضعيفة تحافظ على الخصائص الأساسية للمشتقات الكلاسيكية (مثل خاصية الخطية وقاعدة ليبنز) تحت فرضيات بسيطة.

الكلمات المفتاحية: الدوال الناعمة، دوال الاختبار، المشتقة الضعيفة، فضاءات سوبوليف، الاستمرارية المطلقة، دوال ليبشتز.

Introduction

In classical calculus, the derivative is a local concept; it is defined through the behavior of a function u at a specific point x_0 within a domain U. We say that u is differentiable in U if its derivative exists at each point $x \in U$. On the other hand, weak derivative is a global concept. Instead of focusing on individual points, it defines differentiation in an integral sense rather than the classical pointwise approach.

This approach relies on more general notion of "weakness," which describes properties that are not required to hold pointwise but are instead satisfied when tested against suitable test functions. This fundamental idea enables us to extend the concept of differentiation to a larger class of functions and serves as the basis for the theory of Sobolev spaces.

The theory of Sobolev spaces have been discussed in many classical and modern references, among which the most widely used are [1-4].

One reason for studying the weak derivative is that, in many physical applications, pointwise behavior is less important than integral behavior over small regions. Another reason is that the classical derivative requires strong regularity assumptions; the function must be continuous and sufficiently smooth, and it may fail to exist even for simple continuous functions. For example, the absolute value function is not differentiable at zero, whereas its weak derivative is the sign function. The weak derivative has been widely studied in the literature; see, for instance, [1-5].

This paper is organized as follows. In Section 2, we define test functions and L^p-spaces, and study some of their useful properties that will be used later. In Section 3, we introduce the theory of weak derivatives and show that the weak derivative is a natural generalization of the classical derivative, since both notions coincide for continuously differentiable functions. Moreover, we show that the weak derivative is unique up to a set of measure zero.

Section 4 focuses on various examples of functions that admit weak derivatives, as well as others that do not. In particular, these examples show that functions which fail to be differentiable at a single point may still admit weak derivatives, as in the case of the absolute value function. On the other hand, some functions may fail to have weak derivatives even if they possess classical derivatives almost everywhere; this is typically due to the presence of jump discontinuities at certain points.

Furthermore, Section 5 is devoted to Sobolev spaces, which consist of all functions in L^p- functions whose weak derivatives also belong to L^p. It is shown that Sobolev spaces are Banach spaces. Finally, in Section 6, we show that, under mild assumptions, weak derivatives satisfy most of the important properties enjoyed by classical derivatives.

2. Notation and Preliminaries.

We begin by recalling some basic definitions and well-known results from literature. For more details, see [6-9].

2.1 Definition (The partial derivatives). **[8]**

Let $\Omega \subset \mathbb{R}^n$ be open. Assume $u\colon \Omega \to \mathbb{R}$, $x \in \Omega$. The partial derivative of u in the direction of x_i is defined by

$$\frac{\partial u}{\partial x_i}(x) = \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h},$$

provided the limit exists. We often use shorthand notation such as

$$u_{x_i} = \frac{\partial u}{\partial x_i}, \qquad u_{x_i x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}, \quad \text{etc.}$$

Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$. This vector is called multi-index and its order is defined as

$$|\alpha| = \alpha_1 + \cdots \cdots + \alpha_n$$

For each multi-index α, we define the corresponding derivative of order $|\alpha|$ by

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

For any integer $k \in \mathbb{N}$, we denote by

$$D^k u = \{D^\alpha u \colon |\alpha| = k\},$$

the set of all partial derivatives of order k. If $k = 1$, then

$$Du = \nabla u = (u_{x_1}, u_{x_2}, \cdots, u_{x_n})$$

Let $C^k(\Omega)$ denote the space of all functions that has continuous derivatives of order k.

2.2 Definition (Lipschitz function). **[7]**

Let $A \in \mathbb{R}^n$ and $0 < L < \infty$. A function $f: A \to \mathbb{R}$ is called Lipschitz continuous with constant L, or an L-Lipschitz function, if

$$|f(x) - f(y)| \leq L|x - y|$$

for every $x, y \in \mathbb{R}^n$.

All Lipschitz functions are continuous; but they are not necessarily differentiable, as illustrated by the absolute value function.

2.3 Definition (Smooth function). **[9]**

A function $\varphi: \mathbb{R}^n \to \mathbb{R}$ is said to be smooth or infinitely differentiable if its derivatives of all order exist and are continuous.

The set of all smooth functions in $\Omega \subset \mathbb{R}^n$ is denoted by $C^\infty(\Omega)$ or C^∞.

We will need to define the space of smooth functions that vanish on the exterior of a bounded set, called the space of test functions.

2.4 Definition (Test function)**. [6]**

A function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be a test function if it is smooth function with compact support, where the support of a function $f: \mathbb{R}^n \to \mathbb{R}$ is the closure of the set $\{ x \in \mathbb{R}^n : f(x) \neq 0\}$, i.e.,

$$\operatorname{Supp} f = \overline{\{ x \in \mathbb{R}^n : f(x) \neq 0\}}$$

The *space of test functions on Ω*, denoted by $C_c^\infty(\Omega)$.

2.5 Definition (L^p spaces). [3]

Assume that $\Omega \subset \mathbb{R}^n$ is open, and $1 \leq p \leq \infty$. If $f: \Omega \to \mathbb{R}$ is measurable, we define

$$\|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |f|^p dx\right)^{1/p}, & \text{if } \ 1 \leq p < \infty \\ \operatorname*{ess\,sup}_\Omega |f|, & \text{if } \quad p = \infty. \end{cases}$$

where the essential supremum (ess sup) means that we take the supremum of the function except on a set of measure zero.

We define $L^p(\Omega)$ to be the linear space of all measurable functions $f: \Omega \to \mathbb{R}$ for which $\|f\|_{L^p(\Omega)} < \infty$. Then $L^p(\Omega)$ is a Banach space, provided we identify functions which agree almost everywhere.

2.6 Remark.

For a bounded domain Ω and $1 \leq p_1 \leq p_2 < \infty$. Then we have the inclusion

$$L^{p_2}(\Omega) \subset L^{p_1}(\Omega).$$

In particular,

$$L^\infty(\Omega) \subset L^p(\Omega) \ \text{ for all } 1 \leq p \leq \infty.$$

2.7 Definition. (Absolute continuous function)**.[7]**

A function $u: [a, b] \to \mathbb{R}$ is absolutely continuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $a = x_1 < y_1 \leq x_2 < y_2 \leq \cdots \leq x_m \leq y_m = b$ is partition of $[a, b]$ with

$$\sum_{i=1}^{m}(y_i - x_i) < \delta,$$

then

$$\sum_{i=1}^{m}|u(y_i) - u(x_i)| < \varepsilon$$

2.8 Theorem. [7]

A function $u: [a, b] \to \mathbb{R}$ is absolutely continuous if and only if there exists a function $g \in L^1((a, b))$ such that

$$u(x) = u(a) + \int_a^x g(t)dt.$$

By the Lebesgue differentiation theorem $g = u'$ almost everywhere in (a, b).

2.9 Definition (Locally integrable functions). **[8]**

The space of locally integrable functions on Ω denoted by $L^1_{\text{loc}}(\Omega)$ is defined as the set of measurable functions $f: \Omega \to \mathbb{R}$ such that for all compact set $K \subset \Omega$ the following holds

$$\int_K |f(x)|dx < \infty.$$

Constant functions, piecewise continuous functions and continuous functions are locally integrable. Every function $f \in L^p(\Omega)$, $1 \leq p < \infty$ is locally integrable.

2.10 Lemma. [7]

If $f \in L^1_{\text{loc}}(\Omega)$, satisfies $\int_\Omega f\varphi \, dx = 0$ for every $\varphi \in C_c^\infty(\Omega)$, then $f = 0$ almost everywhere in Ω.

proof.

Let $\Omega' \Subset \Omega$ (i.e. Ω' is open and $\overline{\Omega}'$ is a compact subset of Ω). The space $C_c^\infty(\Omega')$ is dense in $L^p(\Omega')$. There exists a sequence of functions $\varphi_j \in C_c^\infty(\Omega')$ such that $|\varphi_j| \leq 2$ and $\varphi_j \to \text{sgn}\, f$ almost everywhere in Ω'. The dominated convergence Theorem then gives that

$$\begin{aligned} 0 = \lim_{j\to\infty} \int_{\Omega'} f\, \varphi_j \, dx = \int_{\Omega'} \lim_{j\to\infty} f\, \varphi_j \, dx = \int_{\Omega'} f \,\text{sgn}\, f \, dx \\ = \int_{\Omega'} |f| dx. \end{aligned}$$

This implies that $f = 0$ almost every where in Ω' for every $\Omega' \Subset \Omega$. Thus $f = 0$ almost everywhere in Ω.

3.Weak Derivatives.

This section introduces the concept of the weak derivative and presents its main properties. Additional background and proofs can be found in [1, 3-5].

Let $u \in C^1(\Omega)$. Then if $\varphi \in C_c^\infty(\Omega)$, we see from the integration by parts formula that,

$$\int_\Omega u\, \varphi_{x_i} \, dx = -\int_\Omega u_{x_i}\varphi \, dx + \int_{\partial\Omega} u\, \varphi \nu^i \, dS, \qquad (i = 1, \cdots, n).$$

Since $\varphi = 0$ on $\partial\Omega$, we obtain

$$\int_{\Omega} u\, \varphi_{x_i}\, dx = -\int_{\Omega} u_{x_i} \varphi\, dx \quad (i = 1, \cdots, n). \tag{1}$$

Now if k is a positive integer, $u \in C^k(\Omega)$, and $\alpha = (\alpha_1, \cdots, \alpha_n)$ is multi-index of order $|\alpha| = \alpha_1 + \cdots\cdots + \alpha_n = k$, then applying equation (1) $|\alpha|$ times we get

$$\int_{\Omega} u\, D^{\alpha}\varphi\, dx = (-1)^{|\alpha|} \int_{\Omega} D^{\alpha}u\, \varphi\, dx, \tag{2}$$

We now turn to analyzing equation (2) and investigate whether it could remain valid even when u is not continuously differentiable up to order α. The left-hand side of (2) is possible whenever u is locally integrable, whereas the expression $D^{\alpha}u$ on the right-hand side may not be well-defined in the classical sense if $u \notin C^k$. To resolve this, we consider whether there exists a locally integrable function v such that (2) holds when $D^{\alpha}u$ is replaced by v.

3.1 Definition (weak derivative). [1]

Let $u, v \in L^1_{\text{loc}}(\Omega)$, and α is multi-index. If for all $\varphi \in C_c^{\infty}(\Omega)$, it holds that

$$\int_{\Omega} u\, D^{\alpha}\varphi\, dx = (-1)^{|\alpha|} \int_{\Omega} v\, \varphi\, dx,$$

then v is called the α^{th}-*weak partial derivative* of u, written $D^{\alpha}u = v$. If there does not exist such a function v, then u does not have a weak α^{th}-partial derivative.

3.2 Remark:

1. If a function u is classically continuously differentiable, then it also has a weak derivative, and in this case the weak derivative coincides with the classical one.
2. Since the Lebesgue integral is not affected by changes in the functions on sets of measure zero. Therefore, the weak derivative is defined only up to a set of measure zero. This means that a function may fail to be classically differentiable at some points, but it can still have a weak derivative.
3. The behavior of u and v close to the boundary of Ω does not affect the result, since test functions are zero near the boundary.

3.3 Lemma. (uniqueness of weak derivatives**) [3]**

A weak αth partial derivative of u, if it exists, is uniquely defined up to set of measure zero.

Proof.

Assume that $v, \tilde{v} \in L^1_{\text{loc}}(\Omega)$ are both weak αth partial derivatives of u, that is

$$\int_{\Omega} u\, D^{\alpha}\varphi\, dx = (-1)^{|\alpha|} \int_{\Omega} v\, \varphi\, dx = (-1)^{|\alpha|} \int_{\Omega} \tilde{v}\, \varphi\, dx$$

for every $\varphi \in C_c^{\infty}(\Omega)$. This implies that

$$\int_{\Omega} (v - \tilde{v})\varphi\, dx = 0, \qquad \text{for every } \varphi \in C_c^{\infty}(\Omega).$$

Hence, Lemma 2.10 shows that $v = \tilde{v}$ almost everywhere in Ω.

4. Examples:

Example 1. Let $n = 1,$ and $u(x) = |x|$. Then the weak derivative of u is

$$u'(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

To verify this, we compute

$$\int_{-\infty}^{\infty} u\,\varphi' dx = \int_{-\infty}^{0} -x\,\varphi'\,dx + \int_{0}^{\infty} x\,\varphi' dx$$
$$= \int_{-\infty}^{0} \varphi\,dx - \int_{0}^{\infty} \varphi\,dx = -\int_{-\infty}^{\infty} u'\varphi\,dx,$$

Remark: We can choose $u'(0)$ to be any real number, since its value at a single point does not affect the integral.

Example 2. Let $n = 1$ and $\Omega = (0,2)$. Consider

$$u(x) = \begin{cases} x, & 0 < x < 1 \\ \\ 1, & 1 \le x < 2 \end{cases}$$

and

$$v = \begin{cases} 1, & 0 < x < 1 \\ \\ 0, & 1 \le x < 2 \end{cases}$$

Then $u' = v$ in the weak sense. To see this, we show that

$$\int_0^2 u(x)\,\varphi'(x)dx = -\int_0^2 v(x)\varphi(x)\,dx$$

for every $\varphi \in C_c^\infty(\Omega)$.

$$\int_0^2 u(x)\,\varphi'(x)dx = \int_0^1 x\,\varphi'(x)dx + \int_1^2 \varphi'(x)dx$$
$$= x\,\varphi(x)|_0^1 - \int_0^1 \varphi(x)dx + \varphi(2) - \varphi(1)$$
$$= \varphi(1) - \int_0^1 \varphi(x)dx - \varphi(1)$$
$$= -\int_0^1 \varphi(x)dx \quad = -\int_0^2 v(x)\,\varphi(x)dx$$

The following example shows that a function may have a classical derivative almost everywhere but it does not admit a weak derivative.

Example 3. Let $n = 1$ and $\Omega = (0,2)$. Consider

$$u(x) = \begin{cases} x, & 0 < x < 1 \\ \\ 2, & 1 \le x < 2 \end{cases}$$

To show that u does not have a weak derivative we have to show that there does not exist any function $v \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_0^2 u(x)\varphi'(x)dx = -\int_0^2 v(x)\varphi(x)dx\,, \tag{3}$$

for every $\varphi \in C_c^\infty(\Omega)$. Assume that there exists a $v \in L^1_{\text{loc}}(\Omega)$ satisfying (3). Then

$$\begin{aligned} -\int_0^2 v(x)\varphi(x)dx &= \int_0^2 u(x)\varphi'(x)dx \\ &= \int_0^1 x\,\varphi'(x)dx + 2\int_1^2 \varphi'(x)dx \\ &= x\,\varphi(x)|_0^1 - \int_0^1 \varphi(x)dx + 2(\varphi(2) - \varphi(1) \\ &= -\varphi(1) - \int_0^1 \varphi(x)dx \end{aligned} \tag{4}$$

is valid for all $\varphi \in C_c^\infty(\Omega)$. We choose a sequence $\{\varphi_m\}_{m=1}^\infty$ of a smooth function satisfying $0 \le \varphi_m \le 1$, $\varphi_m(1) = 1$ and $\varphi_m(x) \to 0$ for all $x \neq 1$. Replacing φ by φ_m in (4) yields

$$1 = \varphi_m(1) = \int_0^2 v(x)\,\varphi_m(x)\,dx - \int_0^1 \varphi_m(x)dx.$$

Taking the limit for $m \to \infty$ we get

$$1 = \lim_{m\to\infty} \varphi_m(1) = \lim_{m\to\infty} \left[\int_0^2 v(x)\,\varphi_m(x)\,dx - \int_0^1 \varphi_m(x)dx\right] = 0.$$

Hence u can not have a weak derivative.

5. The Sobolev space $W^{k,p}(\Omega)$.

The Sobolev space $W^{k,p}(\Omega)$ consists of functions in $L^p(\Omega)$ that have partial weak derivatives up to order k and they belong to $L^p(\Omega)$. Standard references for Sobolev spaces include [1-4].

5.1 Definition (Sobolev spaces). **[2]**

Assume that Ω is open subset of $\mathbb{R}^n$, and $1 \le p < \infty$. The Sobolev space $W^{k,p}(\Omega)$ consists of functions $u \in L^p(\Omega)$ such that for every multi-index α with $|\alpha| \le k$, the weak derivatives $D^\alpha u$ exist and $D^\alpha u \in L^p(\Omega)$. Thus

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), |\alpha| \le k\}.$$

If $u \in W^{k,p}(\Omega)$, we define its norm to be

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p dx\right)^{1/p}$$

where $D^0 u = D^{(0,0,\cdots,0)}u = u$.

A function $u \in W^{k,p}_{loc}(\Omega)$, if $u \in W^{k,p}(\Omega')$ for every $\Omega' \Subset \Omega$.

5.2 Definition. [3]

Let $\{u_m\}_{m=1}^{\infty}, u \in W^{k,p}(\Omega)$. We say u_m converges to u in $W^{k,p}(\Omega)$, written

$$u_m \to u \quad \text{in} \in W^{k,p}(\Omega),$$

provided

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0.$$

We write

$$u_m \to u \quad \text{in } W^{p}_{\text{loc}}(\Omega),$$

to mean

$$u_m \to u \quad \text{in} \in W^{k,p}(\mathrm{V})$$

for each $\mathrm{V} \Subset \Omega$.

5.3 Definition. **[4]**

we denote by $W_0^{k,p}(\Omega)$ the closure of $C_c^{\infty}(\Omega)$ in $W^{k,p}(\Omega)$. Thus $v \in W_0^{k,p}(\Omega)$ if and only if there exist functions $v_m \in C_c^{\infty}(\Omega)$ such that $v_m \to v \quad \text{in} \in W^{k,p}(\Omega)$.

Note that, functions in $W_0^{k,p}(\Omega)$ are understood to vanish on the boundary of Ω.

5.4 Theorem. [1]

For each $k = 1,2,\cdots$ and $1 \le p < \infty$, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space.

Proof.

1. $\|u\|_{W^{k,p}(\Omega)} \ge 0.$

2. If $u = 0$ then trivially $\|u\|_{W^{k,p}(\Omega)} = 0$. On the other hand, if $\|u\|_{W^{k,p}(\Omega)} = 0$, thus in particular we have $\|u\|_{L^p(\Omega)} = 0$ which implies $u = 0$ a.e.

3. Take $\lambda \in \mathbb{R}$, then

$$\|\lambda u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\le k} \|D^{\alpha}(\lambda u)\|^p_{L^p(\Omega)}\right)^{1/p} = |\lambda| \left(\sum_{|\alpha|\le k} \|D^{\alpha} u\|^p_{L^p(\Omega)}\right)^{1/p} = |\lambda|\|u\|_{W^{k,p}(\Omega)}.$$

4. Assume that $u, v \in W^{k,p}(\Omega)$. Then if $1 \le p < \infty$, Minkowski's inequality implies

$$\|u+v\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\le k} \|D^{\alpha}u + D^{\alpha}v\|^p_{L^p(\Omega)}\right)^{1/p} \le \left(\sum_{|\alpha|\le k} \left(\|D^{\alpha}u\|_{L^p(\Omega)} + \|D^{\alpha}v\|_{L^p(\Omega)}\right)^p\right)^{1/p}$$

$$\le \left(\sum_{|\alpha|\le k} \|D^{\alpha}u\|^p_{L^p(\Omega)}\right)^{1/p} + \left(\sum_{|\alpha|\le k} \|D^{\alpha}v\|^p_{L^p(\Omega)}\right)^{1/p} = \|u\|_{W^{k,p}(\Omega)} + \|v\|_{W^{k,p}(\Omega)}$$

It remains to show that $W^{k,p}(\Omega)$ is complete. Assume that $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $W^{k,p}(\Omega)$. Then for each $|\alpha| \le k$, $\{D^{\alpha}u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $L^p(\Omega)$. Since $L^p(\Omega)$ is complete, there exists functions $u_\alpha \in L^p(\Omega)$ such that

$$D^{\alpha}u_m \to u_\alpha \quad \text{in} \quad L^p(\Omega)$$

for each $|\alpha| \leq k$. In particular,

$$u_m \to u_{(0,\cdots,0)} = u \text{ in } L^p(\Omega).$$

We now claim

$$u \in W^{k,p}(\Omega),\ D^\alpha u = u_\alpha \quad (|\alpha| \leq k).$$

To verify this, fix $\varphi \in C_c^\infty(\Omega)$. Then

$$\int_\Omega u\, D^\alpha\, \varphi\, dx = \lim_{m\to\infty} \int_\Omega u_m D^\alpha \varphi\, dx = \lim_{m\to\infty} (-1)^{|\alpha|} \int_\Omega D^\alpha u_m \varphi\, dx = (-1)^{|\alpha|} \int_\Omega u_\alpha \varphi dx.$$

Hence $D^\alpha u = u_\alpha$ and $u \in W^{k,p}(\Omega)$. Since therefore $D^\alpha u_m \to D^\alpha u$ in $L^p(\Omega)$ for all $|\alpha| \leq k$, we see that $u_m \to u$ in $W^{k,p}(\Omega)$, as required.

6. Properties of weak derivatives and $W^{k,p}(\Omega)$.

The weak derivative retains the most of the properties satisfied by the classical derivative, as satisfied in the following theorem.

6.1 Theorem.[3]

Assume that u, v have weak derivatives of order k and $|\alpha| \leq k$. Then

1. $D^\beta(D^\alpha u) = D^\alpha(D^\beta u)$ for all multi-indices α, β with $|\alpha| + |\beta| \leq k$,
2. for every $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v$ has a weak derivative of order k and
$$D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v,$$
3. (Leibniz's formula) if $\eta \in C_c^\infty(\Omega)$, then ηu has a weak derivative of order k and
$$D^\alpha(\eta u) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta \eta D^{\alpha-\beta} u,$$
where
$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!\,(\alpha-\beta)!}, \qquad \alpha! = \alpha_1!\,\alpha_2! \cdots \alpha_n!$$
and $\beta \leq \alpha$ means that $\beta_j \leq \alpha_j$ for every $j = 1,2,\cdots,n$.

Proof.

1. Let $\varphi \in C_c^\infty(\Omega)$. Then $D^\beta \varphi \in C_c^\infty(\Omega)$. Therefore,
$$(-1)^{|\beta|} \int_\Omega D^\beta(D^\alpha u)\varphi\, dx = \int_\Omega D^\alpha u\, D^\beta \varphi\, dx = (-1)^{|\alpha|} \int_\Omega u\, D^{\alpha+\beta} \varphi\, dx$$
$$= (-1)^{|\alpha|}(-1)^{|\alpha+\beta|} \int_\Omega D^{\alpha+\beta} u\varphi\, dx = (-1)^{|\beta|} \int_\Omega D^{\alpha+\beta} u\varphi\, dx$$
for all test all test functions $\varphi \in C_c^\infty(\Omega)$. Notice that, we get the last equality because
$$|\alpha| + |\alpha+\beta| = \alpha_1 + \alpha_2 + \cdots \alpha_n + (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2) + \cdots + (\alpha_n + \beta_n)$$
$$= 2(\alpha_1 + \alpha_2 + \cdots \alpha_n) + (\beta_1 + \beta_2 + \cdots + \beta_n) = 2|\alpha| + |\beta|.$$
As $2|\alpha|$ is an even number and since the weak derivative is unique, the above estimate implies that $D^\beta(D^\alpha u) = D^\alpha(D^\beta u)$.
2. Follows directly from the definition of weak derivative.
3. First, we consider the case when $|\alpha| = 1$. Let $\varphi \in C_c^\infty(\Omega)$. By Leibniz's rule for differentiable functions and the definition of weak derivative, we get

$$\int_{\Omega} \eta u\, D^{\alpha}\varphi\, dx = \int_{\Omega} (u\, D^{\alpha}(\eta\varphi) - u(D^{\alpha}\eta)\varphi)\, dx = -\int_{\Omega} (\eta\varphi D^{\alpha}u + u(D^{\alpha}\eta)\varphi)\, dx$$
$$= -\int_{\Omega} (\eta D^{\alpha}u + u(D^{\alpha}\eta))\varphi\, dx$$

for all $\varphi \in C_c^{\infty}(\Omega)$. For $|\alpha| > 1$ follows by induction.

6.2 Lemma. [7]

If Ω is connected, $u \in W_{\text{loc}}^{1,p}(\Omega)$ and $Du = 0$ almost everywhere in Ω, then u is almost everywhere constant in Ω.

The following results are collection of the most important properties of Sobolev spaces.

6.3 Theorem (The composition rule) [2]

Let $\Omega \subset \mathbb{R}^n$ be open and $1 \le p \le \infty$. Let $g \in C^1(\mathbb{R})$ satisfies $g(0) = 0$ and $\|g'\|_{L^{\infty}} = M < \infty$, and let $f \in W^{1,p}(\Omega)$. Then the composition $g \circ f$ belongs to $W^{1,p}(\Omega)$ and

$$D_{x_j}(g \circ f) = (g' \circ f).D_{x_j}f$$

Proof.

For $x \in \Omega$ and $f \in W^{1,p}(\Omega)$,

$$|g(f(x))| = |g(f(x)) - g(0)| \le M|f(x) - 0| = M|f(x)|,$$

and therefore $g \circ f \in L^p(\Omega)$. Moreover, $\left|(g' \circ f).D_{x_j}f\right| \le M\left|D_{x_j}f\right|$, and hence $(g' \circ f).D_{x_j}f \in L^p(\Omega)$. Since $f \in W^{1,p}(\Omega)$, then there exists a sequence $\{f_m\}_{m=1}^{\infty}$ in $C_c^{\infty}(\Omega)$

such that $f_m \to f$ in $L^p(\Omega)$ and that $\nabla f_m \to \nabla f$ in $L^p(\Omega)$. Let $\varphi \in C_c^{\infty}(\Omega)$ and $j \in \{1,2,\cdots,n\}$. Then for every $m \in \mathbb{N}$,

$$\int_{\Omega} (g \circ f_m)D_{x_j}\varphi\, dx = -\int_{\Omega} (g' \circ f_m)D_{x_j}f_m\, \varphi dx.$$

Letting $m \to \infty$, we thus obtain

$$\int_{\Omega} (g \circ f)D_{x_j}\varphi\, dx = -\int_{\Omega} (g' \circ f)D_{x_j}f\, \varphi dx,$$

from here and the above estimates follows that $f \in W^{1,p}(\Omega)$.

In contrast of C^1, the following theorem shows that the Sobolev space $W^{1,p}(\Omega)$ is closed under taking absolute values.

6.4 Theorem. [7]

If $u \in W^{1,p}(\Omega)$, then $u^+ = \max\{u, 0\} \in W^{1,p}(\Omega)$, $u^- = -\min\{u, 0\} \in W^{1,p}(\Omega)$, $|u| \in W^{1,p}(\Omega)$ and

$$Du^+ = \begin{cases} Du & \text{almost every where in } \{x \in \Omega: u(x) > 0\}, \\ 0 & \text{almost every where in } \{x \in \Omega: u(x) \le 0\}, \end{cases}$$

$$Du^- = \begin{cases} 0 & \text{almost every where in } \{x \in \Omega: u(x) \ge 0\}, \\ -Du & \text{almost every where in } \{x \in \Omega: u(x) < 0\}, \end{cases}$$

and

$$D|u| = \begin{cases} Du & \text{almost every where in } \{x \in \Omega: u(x) > 0\} \\ 0 & \text{almost every where in } \{x \in \Omega: u(x) = 0\} \\ -Du & \text{almost every where in } \{x \in \Omega: u(x) < 0\} \end{cases}$$

Proof.

Let $\varepsilon > 0$ and let $f_\varepsilon: \mathbb{R} \to \mathbb{R}$, $f_\varepsilon(t) = \sqrt{t^2 + \varepsilon^2} - \varepsilon$. The function f_ε has the following properties: $f_\varepsilon \in C^1(\mathbb{R})$ and $f_\varepsilon(0) = 0$,

$$\lim_{\varepsilon \to 0} f_\varepsilon(t) = |t| \text{ for every } t \in \mathbb{R},$$

$$(f_\varepsilon)'(t) = \frac{t}{\sqrt{t^2+\varepsilon^2}} \quad \text{for every } t \in \mathbb{R},$$

and $\|(f_\varepsilon)'\|_\infty \leq 1$ for every $\varepsilon > 0$. From Theorem 6.3 we conclude that $f_\varepsilon \circ u \in W^{1,p}(\Omega)$ and

$$\int_\Omega (f_\varepsilon \circ u) D_j\, \varphi\, dx = -\int_\Omega (f_\varepsilon)'(u) D_j u\, \varphi\, dx\,, j = 1, \cdots, n,$$

for every $\varphi \in C_c^\infty(\Omega)$. We note that

$$\lim_{\varepsilon \to 0} (f_\varepsilon)'(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0, \end{cases}$$

and consequently

$$\int_\Omega |u| D_j \varphi\, dx = \lim_{\varepsilon \to 0} \int_\Omega (f_\varepsilon \circ u) D_j\, \varphi\, dx = -\lim_{\varepsilon \to 0} \int_\Omega (f_\varepsilon)'(u) D_j u\, \varphi\, dx = -\int_\Omega D_j |u| \varphi\, dx,$$
$$j = 1, \cdots, n,$$

for every $\varphi \in C_c^\infty(\Omega)$, where $D_j u$ is as in the statement of the theorem.

The other claims follow from formulas

$$u^+ = \frac{1}{2}(u + |u|) \text{ and } u^- = \frac{1}{2}(u - |u|)$$

6.5 Corollary.

If $u, v \in W^{1,p}(\Omega)$, then $\max\{u, v\} \in W^{1,p}(\Omega)$ and $\min\{u, v\} \in W^{1,p}(\Omega)$. Moreover,

$$D \max\{u, v\} = \begin{cases} Du & \text{almost every where in } \{x \in \Omega: u(x) \geq v(x)\}, \\ Dv & \text{almost every where in } \{x \in \Omega: u(x) \leq v(x)\}, \end{cases}$$

and

$$D \min\{u, v\} = \begin{cases} Du & \text{almost every where in } \{x \in \Omega: u(x) \leq v(x)\}, \\ Dv & \text{almost every where in } \{x \in \Omega: u(x) \geq v(x)\}, \end{cases}$$

Proof.

It follows directly from Theorem 6.4 and the equalities

$$\max\{u, v\} = \frac{1}{2}(u + v + |u - v|) \text{ and } \min\{u, v\} = \frac{1}{2}(u + v - |u - v|)$$

6.6 Corollary.

If $u \in W^{1,p}(\Omega)$ and $\lambda \in \mathbb{R}$, then $Du = 0$ almost everywhere in $\{x \in \Omega: u(x) = \lambda\}$.

$$\int_{\Omega} u\, D_{x_j}\, \varphi\, dx = -\int_{\Omega} D_{x_j} u\, \varphi\, dx,$$

The following theorem shows that every Sobolev function can be approximated y smooth functions in the Sobolev norm; that is, smooth functions are dense in $W^{1,p}(\Omega)$. For proof see e.g., [3].

6.7 Theorem. **[3]**

Assume Ω is bounded and $u \in W^{1,p}(\Omega)$ for some $1 \le p < \infty$. Then there exist functions $u_m \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that

$$u_m \to u \quad \text{in} \quad W^{1,p}(\Omega).$$

The following theorem establishes the connection between Lipschitz functions and Sobolev functions.

6.8 Theorem.

A function $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ has a representative that is bounded and Lipschitz continuous if and only if $u \in W^{1,\infty}(\mathbb{R}^n)$.

6.9 Lemma. [7]

If $n = 1$ and Ω is an open interval in $\mathbb{R}$. Then $u \in W^{1,p}(\Omega)$ if and only if u equals a.e. an absolutely continuous function whose derivative (which exists a.e.) and the function itself belong to $L^p(\Omega)$. However such a simple characterization only available for $n = 1$. In general a discontinuous functions and/or unbounded function can belong to a Sobolev space, as the following example shows.

Example.

Let $\Omega = B(0,1)$ be the unit ball in $\mathbb{R}^n$ centered at zero, and

$$u(x) = |x|^{-\alpha} \qquad (x \in \Omega,\ x \ne 0)$$

We notice that $u \notin L^\infty(\Omega)$ and we want to find for which $\alpha > 0$, $1 \le p < \infty$, and $n \ge 1$ the function u belongs to $W^{1,p}(\Omega)$.

Note that u is smooth away from 0, i.e., for x with $|x| > 0$ we have that $x \mapsto u(x) \in C^\infty$. Hence in this set we compute the derivatives in the classical sense. We have

$$D_{x_j} u = (-\alpha)|x|^{-\alpha-1} D_{x_j}(|x|) = (-\alpha)|x|^{-\alpha-1} D_{x_j}\left(\left(\sum_{i=1}^{n} x_i^2\right)^{1/2}\right)$$

$$= (-\alpha)|x|^{-\alpha-1}\frac{1}{2}.\frac{2x_j}{|x|} = \frac{-\alpha x_j}{|x|^{\alpha+2}},$$

therefore, for $x \ne 0$, it holds $|\nabla u| = \frac{|-\alpha||x|}{|x|^{\alpha+2}} = \frac{\alpha}{|x|^{\alpha+1}}$.

Let us determine for which values of α we have $u \in L^p(\Omega)$ and $D_{x_j} u \in L^p(\Omega)$. We first define $D_{x_j} u(0) = 0$. Using polar coordinates (in dimension n) we get

$$\int_{\Omega} |u|^p dx = \kappa_n \int_0^1 \rho^{-\alpha p} \rho^{n-1} d\rho = \kappa_n \int_0^1 \rho^{-\alpha p + n - 1} d\rho,$$

where κ_n is the $(n-1)$-measure of the set $\{x \in \mathbb{R}^n : |x| = 1\}$. Thus $u \in L^p(\Omega)$ if and only if $\alpha p < n$ (in particular $u \in L^1(\Omega)$ if and only if $\alpha < n$). Similarly, we get

$$\int_{\Omega} |\nabla u|^p dx = \int_{\Omega} \left(\sum_{j=1}^{n} D_{x_j}^2\right)^{p/2} dx = \alpha^p \kappa_n \int_0^1 \rho^{-(\alpha+1)p+n-1} d\rho,$$

thus $\nabla u \in L^p(\Omega)$ if and only if $(\alpha+1)p < n$ (and $\nabla u \in L^1(\Omega)$ if and only if $(\alpha+1) < n$)

Assume therefore $n \geq 2$ and $\alpha < n-1$, so that $u,\ D_{x_j}u \in L^1(\Omega)$ and we are allowed to consider weak derivatives of u. We want to show that the weak derivative equals the classical derivative $D_{x_j}u$. Let $\varphi \in C_c^{\infty}(\Omega)$, fix $\varepsilon > 0$ and let $B_\varepsilon = B(0,\varepsilon)$. Then, as u is not defined at $x = 0$, we consider the punctured domain $\Omega \backslash B_\varepsilon$. Hence, using integration by parts we get

$$\int_{\Omega \backslash B_\varepsilon} u\, D_{x_j}\, \varphi\, dx = -\int_{\Omega \backslash B_\varepsilon} D_{x_j} u\, \varphi\, dx + \int_{\partial(\Omega \backslash B_\varepsilon)} u\, \varphi\, \nu_j dS \tag{5}$$

Since $\partial(\Omega \backslash B_\varepsilon) = \partial\Omega \cup \partial B_\varepsilon$ and $\varphi = 0$ on $\partial\Omega$. Hence

$$\int_{\partial(\Omega \backslash B_\varepsilon)} u\, \varphi\, \nu_j dS = \int_{\partial B_\varepsilon} u\, \varphi\, \nu_j dS.$$

We estimate the absolute value:

$$\left| \int_{\partial B_\varepsilon} u\, \varphi\, \nu_j dS \right| \leq \int_{\partial B_\varepsilon} |u||\varphi|\, dS.$$

Since $|\varphi| \leq C$ and $|u(x)| = \varepsilon^{-\alpha}$ on ∂B_ε, we get

$$\left| \int_{\partial B_\varepsilon} u\, \varphi\, \nu_j dS \right| \leq \int_{\partial B_\varepsilon} |u||\varphi| dS \leq C \int_{\partial B_\varepsilon} \varepsilon^{-\alpha}\, dS = C\varepsilon^{-\alpha} |\partial B_\varepsilon| \leq C\varepsilon^{n-1-\alpha} \to 0,$$

as $\alpha < n-1$.

Letting $\varepsilon \to 0$ in equation (5) we get

$$\int_{\Omega} u\, D_{x_j}\, \varphi\, dx = -\int_{\Omega} D_{x_j} u\, \varphi\, dx,$$

for all $\varphi \in C_c^{\infty}(\Omega)$. We have thus proved that $D_{x_j}u$ is the weak partial derivative of u.

The conclusion is that $u \in W^{1,p}(\Omega)$ if and only if $\alpha < (n-p)/p$.

Conclusion

In conclusion, this paper provides a clear and comprehensive study of weak derivatives as a generalization of classical derivatives. By using test functions and integral formulations, weak derivatives overcome the limitations of pointwise differentiation while preserving its essential properties, such as linearity and the product rule. We have demonstrated that Sobolev spaces $W^{1,p}(\Omega)$ constructed via weak derivatives, are complete Banach spaces. Furthermore, our analysis clarifies how non-smooth and singular functions can still possess weak derivatives, whereas jump discontinuities prevent their existence. Overall, the concepts presented in this study provide a strong foundation for understanding modern analysis and finding weak solutions to partial differential equations.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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