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Some results on Fixed Points of non-expansive mappings

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بعض النتائج المتعلقة بالنقاط الثابتة للدوال غير التوسعية

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Abstract
This paper investigates the crucial role played by the star-shaped subsets of the underlying domain in guaranteeing the existence of fixed points for non-expansive mappings. It presents a comprehensive survey of several fundamental methods and analytical techniques that have been developed for the study of non-expansive mappings and their fixed point properties. Furthermore, the paper highlights the close and profound relationship between the structure of Banach spaces and the applicability of fixed point theorems, demonstrating how specific characteristics of these spaces significantly influence the existence, uniqueness, and behavior of fixed points under non-expansive mappings.

Keywords: Fixed points, star-shaped sets, contraction mappings. non-expansive mappings.

الملخص
تبحث هذه الورقة البحثية في الدور المحوري الذي تلعبه الفئات الجزئية ذات الشكل النجمي للمجال الأساسي في ضمان وجود نقاط ثابتة لدوال غير التوسعية. وتقدم دراسة شاملة لعدد من الطرق الأساسية والتقنيات التحليلية التي طُوّرت لدراسة الدوال غير التوسعية وخصائص نقاطها الثابتة. علاوة على ذلك، تُبرز الورقة العلاقة الوثيقة والعميقة بين بنية فضاءات باناخ وقابلية تطبيق نظريات النقاط الثابتة، موضحةً كيف تؤثر خصائص محددة لهذه الفضاءات بشكل كبير على وجود النقاط الثابتة وتفردها وسلوكها في ظل الدوال غير التوسعية.

الكلمات المفتاحية: النقاط الثابتة ، فئات الشكل النجمي ، الدوال الانكماشية ، الدوال الغير التوسعية.

1.Introduction

The theory of fixed points has become one of the most influential and widely studied areas in modern nonlinear analysis due to its extensive applications in mathematics and many related disciplines. Over the past several decades, fixed point methods have played a central role in the development of both theoretical and applied mathematics. Among the various classes of mappings considered in fixed point theory, non-expansive mappings have attracted considerable attention because of their natural geometric properties and their close relationship with the structure of Banach spaces. Unlike contraction mappings, non-expansive mappings preserve distances in a weaker sense, making the study of their fixed points substantially more challenging. As a result, the existence of fixed points for such mappings often depends on additional geometric properties of the underlying space or the domain on which the mapping is defined. The development of fixed point theory for non-expansive mappings has led to several fundamental results that establish sufficient conditions for the existence of fixed points in Banach spaces. Important concepts such as convexity, weak compactness, uniform convexity, and normal structure have proved to be essential in guaranteeing the validity of these results. These ideas have not only enriched the theoretical framework of functional analysis but have also inspired numerous extensions and generalizations in more abstract settings. In recent years, research on non-expansive mappings has continued to expand, motivated by new applications and by the search for broader conditions under which fixed points exist , see [1] , [5] , [8] , [10] and [12] .

2. Preliminaries

In this section , we present some standard definitions, illustrative examples , related remarks and well-known results that will be used throughout the remainder of this paper.

Definition 2.1. Let Xbe a linear space over the field of scalars $K$and let E be a subset of X.

(i) Then E is called *star-shaped with respect to a point* $x_0 \in E$ if for every $x \in E$and $t \in [0,1]$, then $(1-t)x_0 + tx \in E$.

If $x_0 = 0$, we say that $E$is *star-shaped with respect to the origin* .

(ii) Then E is called a *convex set* if for every $x, y \in E$and $t \in [0,1]$, then

$$(1-t)x + ty \in E.$$

Clearly, every convex set is star-shaped , however, the converse is not necessarily true .

Definition 2.2. Let Xbe a linear space over the field of scalars $K$and let $\|.\| : X \to \mathbb{R}$ be a mapping such that

(i) $\| x \| \geq 0$,

(ii) $\| x \| = 0$ if and only if $x = 0$,

(iii) $\| \alpha x\| = | \alpha | \| x\|$,

(iv) $\| x + y\| \leq \| x \| + \| y\|$,

for all $x, y \in X$ and $\alpha \in K$.

Then the mapping $\|.\|$ is called a *norm* on $X$and the pair $(X, \| . \|)$ is called a *linear normed space*, or briefly *a normed space.*

Definition 2.3. *Let* $(X, \| . \|)$ be a normed space. A subset E of $X$is called *bounded* if there exists a positive constant M such that $\| x \| \leq M$ for all $x \in E$,

or equivalent,E is a bounde*d* set if it contained in a ball $B(0, M)$ centered at 0, where

8

$$B(0, M) = \{x \in X: \| x \| \leq M\}.$$

Definition 2.4. Let $(X, \| . \|)$ be a normed space. The sequence $\{ x_n \}$ in $X$is said to *converge* to the point x in Xif for each $\varepsilon > 0$, there exists a positive integer N such that

$$\| x_n - x \| < \varepsilon \quad \text{for all } n > N,$$

Or $\| x_n - x\| \to 0$ as $n \to \infty$.

The point x is called the *limit* of the sequence $\{ x_n \}$ and we write $\lim_{n \to \infty} x_n = x$.

It is denoted by $x_n \to x$ as $n \to \infty$.

Definition 2.5. A subset E of a normed space $(X, \| . \|)$ is called a *closed set* in X if E contains all of its limit points .

That is , if $x_n \in E$ and $x_n \to x$ as $n \to \infty$, then $x \in E$.

Remark. Let T be a mapping from E into itself and let I be the identity mapping . Then

$$(I - T) (E) = \{ (I - T) (x) : x \in E\}.$$

If E is a closed set , then $(I - T) (E)$ need not be closed .

For example , let $E = \mathbb{R}$ which is a closed set .

Define a function $T: \mathbb{R} \to \mathbb{R}$ by

$$T(x) = x - e^x \ (x \in \mathbb{R}).$$

Then $(I - T) (x) = x - (x - e^x) = e^x$.

So $(I - T) (E) = \{ e^x : x \in \mathbb{R}\} = (0, \infty)$.

But $(0, \infty)$ is not closed in $\mathbb{R}$. . .

Definition 2.6. Let $(X, \| . \|)$ be a normed space. Let T be a mapping from X into itself. Then T is said to be a *continuous mapping* if any sequence $\{ x_n \}$ in X such that $x_n \longrightarrow x$ implies that $T(x_n) \longrightarrow T(x)$.

That is, if $\| x_n - x\| \longrightarrow 0$ as $n \to \infty$ implies that

$$\| T(x_n) - T(x)\| \longrightarrow 0 \text{ as } n \to \infty.$$

Remark. Let T be a continuous mapping . If $\lim_{n \to \infty} x_n = x$, then

$$T(x) = \lim_{n \to \infty} T(x_n),$$

more obviously , T is a continuous mapping if $\lim_{n \to \infty} T(x_n) = T(\lim_{n \to \infty} x_n)$.

Definition 2.7. *Let* $(X, \| . \|)$ be a normed space. A sequence $\{ x_n \}$ in X is called a *Cauchy sequence* if for each $\varepsilon > 0$, there exists a positive integer N such that

$$\| x_n - x_m \| < \varepsilon \quad \text{for all } n, m > N,$$

or $\| x_n - x_m\| \to 0$ as $n, m \to \infty$.

Definition 2.8. A normed space $(X, \| . \|)$ is called *complete* if every Cauchy sequence in X is convergent to an element in X. A complete normed space $(X, \| . \|)$ is called a *Banach space*.

Theorem 2.9 [4] . *Let $(X, \|.\|)$ be a Banach space and let E be a closed subset of X. Then every Cauchy sequence $\{ x_n \}$ in E converges to a point of E.*

Definition 2.10. A subset E of a Banach space $(X, \|.\|)$ is called *weakly compact* if for every sequence $\{ x_n \} \subseteq E$ has a subsequence$\{ x_{n_k} \}$ that converges weakly to some point $x \in E$.

The weak convergence means that $x_n \longrightarrow x$ if and only if $T(x_n) \longrightarrow T(x)$ for every $T \in X^{\bullet}$, where denotes the dual space of X.

Definition 2.11 . A bounded convex subset E of a Banach space $(X, \|.\|)$ is said to be have *normal structure* if every bounded convex subset $K$of $E$with more than point contains a point $x \in K$ such that $\sup_{y\in K}\| x - y\| < diam(K)$ for some $x \in K$, where $diam(K) = \sup_{x,y\in K}\| x - y\|$.

Definition 2.12. A Banach space $(X, \|.\|)$ is called *uniformly convex* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in X$satisfy $\| x\| \leq 1, \| y\| \leq 1$ and $\| x - y\| \geq \varepsilon$, then $\left\| \frac{x+y}{2}\right\| \leq 1 - \delta$.

Definition 2.13. Let Xbe a linear space over the field of scalars $K$and let $\langle . , . \rangle : X \times X \to K$ be a mapping satisfying the following properties :

(i) $\langle x,x\rangle \geq 0$,

(ii) $\langle x, x\rangle = 0$ if and only if $x = 0$

(iii) $\langle x,y\rangle = \overline{\langle y, x\rangle}$,

(iv) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$,

(v) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,

for all $x, y, z \in X$ and $\alpha \in K$.

Then $\langle \cdot , \cdot\rangle$ is called an *inner product* on $X$and the pair $(X, \langle \cdot , \cdot\rangle)$ is called an *inner product space* .

Definition 2.14. A *Hilbert space* is an inner product space H that **is** complete with respect to the norm induced by its inner product . The norm is defined by :

$$\|x\| = \sqrt{< x, x >} \quad (x \in H).$$

Example 2.15 [7]. Let ℓ^2 denote the linear space of all sequences (x_n) such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$.

Let $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in \ell^2$. The inner product on ℓ^2 is given by

$$\langle x,y\rangle = \sum_{n=1}^{\infty} x_n\overline{y_n},$$

and the induced norm is

$$\| x \| = (\sum_{n=1}^{\infty} |x_n|^2)^{\frac{1}{2}}.$$

Then ℓ^2 is a Hilbert space .

Theorem 2.16 [9]. *Every Hilbert space is uniformly convex Banach space .*

We now proceed to examine the concepts of contraction mappings and non-expansive mappings in greater detail, highlighting their definitions, properties, and significance in the context of fixed point theory.

We recall the following standard definition.

Definition 2.17. Let Xbe a non-empty set and let T be a mapping from Xinto itself. If there exists an element x in X such that $Tx = x,$ then x is called a *fixed point* of T

For any given $x \in X$, we define $T^n(x)$ by $T^{n+1}(x) = T(T^n(x))$.

It clear that if x is a fixed point of T, then x is also a fixed point of $T^n$for every positive integer n. That is , $T^n(x) = x.$

Let $x_0 \in X.$ The iterative sequence $\{x_n\}$ in $X$is given by :

$$x_{n+1} = T(x_n) \quad (n = 0 , 1 , 2 , ...),$$

or

$$x_n = T(x_{n-1}) \quad (n = 1 , 2 , ...).$$

We construct a sequence depending on $x_0$as follows :

$$x_1 = T(x_0)$$

$$x_2 = T(x_1) = T(T(x_0)) = T^2(x_0)$$

$$x_3 = T(x_2) = T(T^2(x_0)) = T^3(x_0).$$

By the same argument, we obtain

$$x_n = T^n(x_0) \quad (n = 1 , 2 , ...).$$

Definition 2.18. Let $(X, \| . \|)$ be a normed space and let E be a subset of X. A mapping $T$from Einto itself is called a *contraction mapping* f there exists k with $0 \leq k < 1$ such that

$$\| T(x) - T(y)\| \leq k \| x - y\| \quad \text{for all } x, y \in E.$$

The constant k is called the *contractivity coefficient* .

The following provides a clear example of a contraction mapping .

Example 2.19. Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$T(x) = \frac{1}{5} x + 3.$$

Let $x,\ y\ \in\ \mathbb{R}$. Then

$$\| T(x) - T(y) \| \ = \frac{1}{5} \| x - y \|.$$

Thus T is a contraction mapping with $k = \frac{1}{5}$.

Lemma 2.20. *Let $(X, \ \|.\|)$ be a normed space. Then every contraction mapping from X into itself is continuous.*

Proof . Let T be a contraction mapping from X into itself . Then

$$\| T(x) - T(y) \| \ \leq \mathrm{k} \| x - y \| \quad \text{for all} \quad x,\ y \in \ X,$$

where $0 \ \leq \ k < \ 1$.

Let (x_n) be a sequence in X. Then we have

$$\| T(x_n) \ - \ T(x) \| \ \leq \ \mathrm{k} \| x_n \ - \ x \| \ .$$

Let $x_n \ \longrightarrow \ x$ in X. Then $\| x_n - \ x \ \| \to \ 0$ as $n \ \to \ \infty$.

Therefore $\| T(x_n) - \ T(x) \| \ \to \ 0$ as $n \ \to \ \infty$.

Thus $T(x_n) \longrightarrow T(x)$ and hence T is a continuous mapping .

Lemma 2.21. *Let $(X, \ \|.\|)$ be a normed space . If T is a contraction mapping from X into itself with k is the contractivity coefficient of T, then T^n is contraction for every positive integer n with k^n, where k^n is the contractivity coefficient of T^n.*

That is , $\| T^n(x) - T^n(y) \| \ \leq \ k^n \| x - \ y \|$ for all $x,\ y \in \ X$.

Proof . The proof is established by applying mathematical induction on n.

The following theorem gives the conditions for a contraction mapping to possess a unique fixed point.

Theorem 2.22 [13] . *Let $(X, \ \|.\|)$ be a Banach space and let E be a non-empty , bounded and closed subset of X. If T is a contraction mapping from E into itself , then T has a unique fixed point in E*

Definition 2.23. Let $(X, \ \|.\|)$ be a normed space and let E be a subset of X. A mapping T from E into itself is called a *non-expansive mapping* if

$$\| T(x) - T(y) \| \ \leq \ \| x - y \| \quad \text{for all} \ x,\ y \in \ E.$$

We provide a few examples relating to the non-expansive mappings .

Examples 2.24. (1) Let $T: \ \mathbb{R} \ \to \ \mathbb{R}$ be defined by

$$T(x) \ = \frac{1}{3} \left(x + \sin \ x \right).$$

Let $x,\ y\ \in\ \mathbb{R}$. Then

$$\|\ T(x)-T(y)\| \ = \ |T(x)-T(y)\ |\ .$$

We

have

$$|T(x)-T(y)\ | = \ \left|\frac{1}{3}\ (\ x\ +\sin\ x\ )\ -\frac{1}{3}\ (\ y\ +\sin\ y\ )\right|$$

$$\leq \frac{1}{3}\ |x- \ y| + \frac{1}{3}|\sin\ x\ -\sin\ y|.$$

Since $|\sin\ x\ -\sin\ y|\ \leq\ |x- \ y|$, it follows that

$$|T(x)-T(y)\ |\ \leq \frac{2}{3}\ |x- \ y|$$

$$\leq\ |x- \ y|\ .$$

Thus $T$is a non-expansive mapping .

(2) Let C_0 be the space of all convergent sequences to zero with the following norm :

$$\|\ x\| \ = \ sup \ \ |x_i\ | \ \ (i = 1,2,\dots\ , x \in C_0\ ).$$

Let B be the closed unit ball in C_0 . Define $T\colon\ B \to B$ by

$$T(x_1,x_2,x_3,\dots\ ) = \ (1,x_1,x_2,\dots\ ).$$

Let $x,y \in\ C_0$. Then

$$x = (x_1,x_2,\dots\ x_n,\dots\ ) \to 0$$

and

$$y = (y_1,y_2,\dots\ y_n,\dots\ ) \to 0.$$

Therefore

$$\|\ x-y\| \ = \ sup \ \ |x_i - y_i\ | \ \ (i = 1,2,\dots\ )$$

$$= \ sup \ \ \{|x_1 - y_1\ |,\ |x_2 - y_2\ |,\dots\ \}. \qquad (1.1)$$

We have

$$T(x) = \ (1,x_1,x_2,\dots\ ) \text{ and } \ T(y) = \ (1,y_1,y_2,\dots\ ).$$

Then

$$T(x)-T(y) = \ (0,x_1-y_1,x_2-y_2,\dots\ ),$$

and so

$$\| T(x) - T(y)\| \quad = \quad sup \quad \{0, |x_1 - y_1 \ |, \ |x_2 - y_2 \ |, \dots \ \}$$

$$= \ sup \quad \{|x_1 - y_1 \ |, \ |x_2 - y_2 \ |, \dots \ \}. \qquad (1.2)$$

It follows from (1.1) and (1.2) that

$$\| T(x) - T(y)\| \quad = \ \| x - y\| \ .$$

Thus T is a non-expansive mapping .

We now present some observations concerning the existence and uniqueness of fixed points associated with non-expansive mappings.

Remarks (1) A non-expansive mapping does not necessarily have a fixed point..

For example :

As in example 2.24 (ii) , let B be the closed unit ball in the space C_0 .

Define T: $B \to B$ by

$$T(x_1, x_2, x_2 \ , \dots \ ) = \ (1, x_1, x_2, \dots \ ).$$

The fixed point equation is : .$T(x_1, x_2, x_2 \ , \dots \ ) = \ (x_1, x_2, x_2 \ , \dots \ )$.

Then $(x_1, x_2, x_3, \dots \ ) = \ (1, x_1, x_2, \dots \ )$.

Therefore $x_1 = \ 1, x_1 = x_2 = 1, \ x_2 \ = x_3 = 1, \dots$

So we have $(x_1, x_2, x_3, \dots \ ) = \ (1,1,1, \dots \ ) \notin \ C_0$. .

Thus $T$is a non-expansive mapping but Thas no a fixed point in C_0.

(2) A fixed point of a non-expansive mapping on a Banach space need not be unique

For example :

Let $X = \ C[0,1]$ be the space of real – valued continuous functions on the closed interval [0,1] equipped with the supremum norm :

$$\| f\|_\infty = \sup_{x\in[0,1]} | f(x)|.$$

Define

$$(Tf)(x) = \ \int_0^1 \quad f(t)dt.$$

Let $f, g \ C[0,1]$. Then

$$\| \mathrm{T}f - Tg \ \|_\infty = \sup_{x\in[0,1]} \int_0^1 \quad |f(t) - g(t)|dt$$

$$\leq \int_0^1 \sup_{x\in[0,1]} |f(t) - g(t)|dt$$

$$= \|f - g\|_\infty.$$

Thus T is a non-expansive mapping .

The fixed point equation is $f(x) = \int_0^1 f(t)dt.$

Therefore every constant function is a fixed point . Thus the fixed point is not unique.

Lemma 2.25. *Let $(X, \|.\|)$ be a normed space. Then every non-expansive mapping from X into itself is continuous.*

Proof . The result follows by an argument similar to the proof of Lemma 2.20.

Lemma 2.26. *Let $(X, \|.\|)$ be a normed space . If T is a non-expansive mapping from X into itself , then T^n is non-expansive mapping from X into itself for every positive integer n.*

Proof . The proof proceeds by induction on n.

3. Main Results

In this section , we introduce the main results of the present work that are related to non-expansive mappings. We provide a detailed exposition of these results, including their proofs, underlying assumptions and mathematical consequences.

Proposition 3.1. *Let $(X, \|.\|)$ be a normed space and let E be a non-empty subset of X. Let T be a contraction mapping from X into E with constant $0 \leq k < 1$ and P be a non -expansive mapping from E onto X. Then PT is a contraction mapping from X into itself with the same constant $0 \leq k < 1$.*

Proof . Let T be a contraction mapping from X into E. Then

$$\|T(x) - T(y)\| \leq k\|x - y\| \quad \text{for all} \quad x, y \in X,$$

where $0 \leq k < 1$.

Let P be a non-expansive mapping from E into X. Then

$$\|P(x) - P(y)\| \leq \|x - y\| \quad \text{for all} \quad x, y \in E.$$

Define $(PT)(x) = P(T(x))$ for all x.

Then $\|(PT)(x) - (PT)(y)\| = \|P(T(x)) - P(T(y))\|$

$$\leq \|T(x) - T(y)\|$$

$$\leq k\|x - y\|.$$

Thus $\| (PT)(x) - (PT)(y)\| \leq k\| x - y\|$.

Hence PT is a contraction mapping with constant $0 \leq k < 1$.

Corollary 3.2. *Let $(X, \| . \|)$ be a normed space and let E be a non-empty subset of X. Let T be a contraction mapping from X into E and P be a non-expansive mapping from E onto X. Then $(PT)^n$ is a contraction mapping from X into itself for every positive integer n.*

Proof . Let T be a contraction mapping from X into E and P be a non-expansive mapping from E onto X. It follows from Proposition 3.1 that PT is a contraction mapping . Thus $(PT)^n$ is a contraction mapping (Lemma 2.21) .

Remark. In view of Lemma 2.20, the mapping $(PT)^n$ is continuous.

Theorem 3.3. *Let $(X, \| . \|)$ be a Banach space and let E be a nonempty bounded and closed subset of X. Let T be a contraction mapping from X into E with constant $0 \leq k < 1$ and P be a non-expansive mapping from E onto X such that P is a retraction mapping ($P(x) = x$ for all $x \in E$). If a sequence $\{ x_n\}$ in E is defined as follows :*

$$x_n = (PT)^n(x_0) \quad (x_0 \in E),$$

then $\{ x_n\}$ has a unique fixed point in E.

Proof . Let T be a contraction mapping from X into E. Then

$$\|T(x) - T(y)\| \leq k\| x - y\| \quad \text{for all} \quad x, y \in X,$$

where $0 \leq k < 1$.

Let $x_0, x_1 \in E$. Then

$$\begin{aligned} \| (PT)^n(x_0) - (PT)^n(x_1) \| &= \|PT((PT)^{n-1}(x_0)) - PT((PT)^{n-1}(x_1))\| \\ &\leq k\| (PT)^{n-1}(x_0) - (PT)^{n-1}(x_1)\| \\ &\leq k^2\| (PT)^{n-2}(x_0) - (PT)^{n-2}(x_1)\| \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq k^n\|x_0 - x_1\| . \end{aligned}$$

Thus

$$\| (PT)^n(x_0) - (PT)^n(x_1) \| \leq k^n\|x_0 - x_1\| \tag{3.1}$$

Now , we show $\{ x_n \}$ is a Cauchy sequence in E.

For any $x_0 \in E$, the sequence $\{ x_n \}$ is defined as follows ::

$$x_n = (PT)^n(x_0),$$

Let $m > 0$ and $n > 0$ with $m > n$, take $m = n + P$.

It follows from (3.1) that

$$\| x_n - x_m \| = \| x_n - x_{n+p} \|$$

$$= \| (x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \dots + (x_{n+p-1} - x_{n+p}) \|$$

$$\leq \| x_n - x_{n+1}\| + \| x_{n+1} - x_{n+2} \| + \dots + \| x_{n+p-1} - x_{n+p}\|$$

$$= \| (PT)^n(x_0) - (PT)^n(x_1) \| + \| (PT)^{n+1}(x_0) - (PT)^{n+1}(x_1) \| +$$

$$\dots + \| (PT)^{n+p-1}(x_0) - (PT)^{n+p-1}(x_1) \|$$

$$\leq k^n\| x_0 - x_1\| + k^{n+1}\| x_0 - x_1\| + \dots + k^{n+p-1}\| x_0 - x_1\|$$

$$= (k^n + k^{n+1} + \dots + k^{n+p-1})\| x_0 - x_1\|$$

$$= k^n(1 + k + \dots + k^{p-1})\| x_0 - x_1\|$$

$$\leq \frac{k^n}{1-k} \| x_1 - x_0 \| \qquad (3.2)$$

Since E is bounded , so there exists $M > 0$ such that

$$\| x_1 - x_0\| \leq M \quad (x_1 - x_0 \in E).$$

Then the inequality (3.2) becomes $\| x_n - x_m \| \leq \frac{k^n}{1-k} M$.

Since $0 \leq k < 1$, so we have $k^n \to 0$ as $n \to \infty$.

Therefore $\| x_n - x_m \| \to 0$ as $n, m \to 0$.

Hence $\{ x_n \}$ is a Cauchy sequence in E.

Since E is a closed subset of a Banach space X, so the Cauchy sequence $\{ x_n \}$ is convergent to an element in E (Theorem 2.9) , say x.

That is , $\lim_{n \to \infty} x_n = x$. It follows that $\lim_{n \to \infty} x_{n+1} = x$.

By continuity of PT, we have

$$\begin{aligned}(PT)(x) &= P(T(x)) \\ &= P(T(\lim_{n \to \infty} x_n)) \\ &= \lim_{n \to \infty} P(T(x_n)) \\ &= \lim_{n \to \infty} P(x_{n+1}) \\ &= \lim_{n \to \infty} x_{n+1} \\ &= x.\end{aligned}$$

Therefore $(PT)(x) = x$. Thus x is a fixed point of PT.

Now, we have to prove that such a fixed point is unique .

Let x be another fixed point of PT That is , $(PT)(x) = x$.

We have

$$\| x - x \| = \| (PT)(x) - (P\ T)(x) \|$$

Since PT is a contraction mapping , so

$$\| (PT)(x) - (P\ T)(x) \| \leq k \| x - x \|,$$

with $0 \leq k < 1$.

So we obtain $\| x - x \| \leq k \| x - x \|$.

Therefore $(1 - k) \| x - x \| \leq 0$.

Since $1 - k > 0$, it follows that $\| x - x \| = 0$ and so $x = x'$.

Hence $x$is a unique fixed point of PT.

The following classical and well-established results constitute a fundamental basis for the development of fixed point theory for non-expansive mappings. They provide a variety of important conditions and assumptions under which non-expansive mappings are guaranteed to possess fixed points.

Theorem 3.4 (Kirk) [11] **.** *Let $(X, \|.\|)$ be a Banach space and let E be a non-empty weakly compact t, convex subset of X. Assume that E has normal structure. If $T$is a non-expansive mapping from Einto itself , then Thas a fixed point in E.*

Theorem 3.5 (Browder) [6] **.** *Let $(X, \|.\|)$ be a uniformly convex Banach space and let E be a non-empty bounded , closed and convex subset of X. If $T$is a non-expansive mapping from E into itself , then Thas at least one fixed point in E.*

Since every Hilbert space is a uniformly convex Banach space , the Browder theorem immediately yields the following special case :

Theorem 3.6. *Let Xbe a Hilbert space and let E be a non-empty bounded , closed and convex subset of X. If $T$is a non-expansive mapping from Einto itself , then Thas at least one fixed point in E.*

Under the assumptions that E is a nonempty, bounded, closed of a Banach space $X$and E is star-shaped with respect to the origin, and that $(I - T)(E)$is closed in X, we prove in the following theorem that the non-expansive mapping T from Einto itself has a fixed point in E.

Theorem 3.7. *Let* $(X, \|.\|)$ *be a Banach space and let* E *be a non-empty bounded , closed subset of* X. *Let* E *be star–shaped set with respect to the origin. If* T *is a non-expansive mapping from* E *into itself . Assume that* $(I - T)(E)$ *is closed in* X, *then* T *has a fixed point in* E.

Proof . Let E be a star – shaped set with respect to the origin and T be a non-expansive mapping from E into itself . Then for each $x \in E$, we define the mapping $T_n : E \to E$ by

$$T_n(x) = (1 - \frac{1}{n})T(x) \quad (x \in E, n \geq 2).$$

We have

$$\| T_n(x) - T_n(y)\| = \left\| (1 - \frac{1}{n})T(x) . - (1 - \frac{1}{n})T(y)\right\|$$

$$= (1 - \frac{1}{n}) \| T(x) - T(y)\|$$

$$\leq (1 - \frac{1}{n}) \| x - y \| .$$

Thus T_n is a contraction mapping from non-empty bounded , closed subset E of X and hence T_n has a unique fixed point in E (Theorem 2. 22) , say x_n. That is , $T_n(x_n) = x_n$.

Since E is bounded , so E is contained in a ball centered at 0 of radius r.

Therefore $\| T(x_n)\| \leq r$.

We have

$$\| T(x_n) - x_n \| = \| T(x_n) - T_n(x_n) \|$$

$$= \left\|T(x_n) - (1 - \frac{1}{n})T(x_n)\right\|$$

$$= \frac{1}{n} \| T(x_n)\|$$

$$\leq \frac{1}{n} r \to 0 \text{ as } n \to \infty .$$

Since $x_n - T(x_n) \in (I - T)(E)$ and $(I - T)(E)$ is a closed set in X, so the limit of the sequence belongs to $(I - T)(E)$.

That is , $0 \in (I - T)(E)$. Then there exists a point u in E such that $T(u) - u = 0$ and hence $T(u) = u$. Thus u is a fixed point of T.

Remark . The assumptions of the theorem 3.7 ensure the existence of fixed points but do not guarantee their uniqueness.

For example : Let $X = \mathbb{R}^2$ with the usual Euclidean norm is defined by

$$\| (x, y) \| = \sqrt{x^2 + y^2} \text{ , for every } (x, y) \in \mathbb{R}^2.$$

Then $\mathbb{R}^2$a Banach space.

Let $E = \{(x, y): x^2 + y^2 \leq 1\}$.

Then E is a non-empty, bounded, closed, and star-shaped subset of $\mathbb{R}^2$with respect to the origin. Define $T: E \to E$ by

$$T(x, y) = (x, 0).$$

Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$. Then

$$T(u) = (x_1, 0) \text{ and } T(v) = (x_2, 0).$$

Therefore $T(u) - T(v) = (x_1 - x_2, 0)$ and so

$$\|T(u) - T(v)\| = \sqrt{(x_1 - x_2)^2}$$

$$= |x_1 - x_2|.$$

Also , we have $u - v = (x_1 - x_2, y_1 - y_2)$ and therefore

$$\|u - v\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Since $(x_1 - x_2)^2 \leq (x_1 - x_2)^2 + (y_1 - y_2)^2$, taking square roots gives

$$|x_1 - x_2| \leq \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Thus $\|T(u) - T(v)\| \leq \|u - v\|$.

Hence T is a non-expansive mapping from Einto itself .

We have $(I - T)(E) = \{(0, y): |y| \leq 1\}$

Hence $(I - T)(E)$ is a closed set in $\mathbb{R}^2$

The fixed point equation is $T(x, y) = (x, y)$. So $(x, 0) = (x, y)$.

Therefore the set of fixed points is$\{(x, 0): |x| \leq 1\}$, which contains infinitely many fixed points . Thus the fixed point is not unique even though all the hypotheses of Theorem 3.7 are satisfied .

Conclusion

Since the existence of fixed points for non-expansive mappings cannot be guaranteed in general, the characterization of the structure of the underlying space and the properties of the domain are essential. This paper reviews fixed point results for non-expansive mappings and emphasizes the significance of conditions such as boundedness, closedness, convexity, and star-shapedness. These results illustrate the various connections between the structure of Banach spaces and the fixed point theory of non-expansive mappings.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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