



Existence of a Unique Monotone Solutions for a Fractional Differential Equation in L^1 Using a Measure of Noncompactness

Masouda M. A. Al-Fadel ^{1*}, Amna F. B. Al-Hawari ²

¹ Department of Mathematics Faculty of science, University of Derna, Alqubba, Libya

² Department of Mathematics Faculty of science, University of Derna, Derna, Libya

وجود حلول رتيبة وحيدة لمعادلة تفاضلية كسرية في فضاء L^1 باستخدام مقياس عدم التراص

مسعودة الفضيل^{1*}، أمنا الهواري²
¹ قسم الرياضيات، كلية العلوم، جامعة درنة، القبة، ليبيا
² قسم الرياضيات، كلية العلوم، جامعة درنة، درنة، ليبيا

*Corresponding author: massouda.alfadhel@uod.edu.ly

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Abstract:

In this paper, we investigate the existence and uniqueness of monotone solutions for a class of fractional differential equations in the Banach space L^1 . The considered problem involves fractional-order terms with multiple non-independent components, which allows us to formulate the problem in an appropriate integral form. By constructing a suitable operator associated with the corresponding integral equation, we establish the existence and uniqueness of solutions under appropriate assumptions on the nonlinear terms and the fractional operators. The proof is mainly based on the theory of measures of noncompactness and an appropriate fixed-point theorem. Consequently, the existence of a unique monotone and non-decreasing solution in L^1 is established.

Keywords: Fractional calculus, Monotone solutions, Measures of noncompactness, Fixed point theory.

المخلص

نبحث في هذه الورقة مسألة وجود ووحدانية الحلول الرتيبة لفئة من المعادلات التفاضلية الكسرية في فضاء باناخ وتتضمن المسألة المدروسة حدوداً ذات رتبة كسرية مكونة من مركبات متعددة غير مستقلة، مما يتيح لنا صياغة المسألة في صورة تكاملية ملائمة. ومن خلال إنشاء مؤثر مناسب مرتبط بالمعادلة التكاملية المقابلة، نشب وجود ووحدانية الحلول في ظل افتراضات ملائمة تتعلق بالحدود غير الخطية والمؤثرات الكسرية. وتستند عملية الإثبات بشكل أساسي إلى نظرية مقياس عدم التراص ومبرهنة مناسبة للنقطة الثابتة. وبناءً على ذلك، يتم إثبات وجود حل وحيد رتيب وغير متناقص في الفضاء.

الكلمات المفتاحية: الحساب الكسري، الحلول الرتيبة، مقياس عدم التراص، نظرية النقطة الثابتة.

Introduction

In recent decades, fractional calculus has emerged as a powerful mathematical tool due to its ability to describe memory effects and hereditary properties in various physical, biological, and engineering systems. In particular, fractional-order differential equations provide a more realistic modeling framework for complex phenomena than classical integer-order differential equations. Among these formulations, Bagley–Torvik-type problems have attracted significant research interest owing to their extensive applications in real-world phenomena, particularly in modeling the motion of rigid plates submerged in Newtonian fluids and depicting viscoelastic behaviors in materials [11].

The classical Bagley–Torvik equation traditionally combines integer-order derivatives with fractional derivatives of order $1/2$ or $3/2$. Over the years, various fractional operators have been developed and analyzed, including the Caputo, Riemann–Liouville, Hadamard, and Hadamard–Caputo fractional derivatives [1,9]. More recently, generalized fractional derivatives, most notably the Hilfer fractional derivative, have gained substantial attention. The Hilfer operator encompasses both the Caputo and Riemann–Liouville derivatives as special limiting cases, thereby offering a unified and flexible framework to study non-local behavior in differential equations.

Despite the extensive literature on Bagley–Torvik-type equations, analyzing the existence and uniqueness of monotone solutions within non-reflexive Banach spaces, such as L_1 , presents significant analytical challenges due to the lack of compactness properties. To overcome this difficulty, the technique of measures of noncompactness combined with appropriate fixed-point theorems has proven to be an indispensable approach [2,5,7].

In this paper, we establish the existence and uniqueness of a monotone (non-decreasing) solution for a generalized Bagley–Torvik-type fractional differential equation involving the Hilfer fractional derivative within the Banach space $L_1(0,1)$. By converting the boundary/initial value problem into an equivalent nonlinear integral operator, and by applying the measure of noncompactness technique alongside a suitable fixed-point theorem, we obtain rigorous conditions ensuring the existence of a unique non-negative, monotonic solution.

Definition 1:

Bagley-Torvik problem it is a second-order differential equation that contains a fractional derivative and its known formula:

$$aD^2x(t) + bD^{3/2}x(t) + cx(t) = f(t, x(t))$$

Initial conditions

$$x(0) = x_0, \quad x'(0) = x_1$$

where $D^{3/2}$ is a fractional derivative.

It was chosen $\alpha = 3/2$ not random. because this test provides a suitable mathematical framework for analyzing the equation and allows, converting it into an integral equation and studying the existence of a solution for it using the fixed-point theory.

Definition 2:

Let $\alpha \in (1,2), \beta \in [0,1], f \in L_1$ and $(I^{(1-\beta)(2-\alpha)}f)(t) \in AC^2(I)$ The Hilfer fractional derivative of order α and type β of f is defined as:

$$(D^{\alpha,\beta}f)(t) = \left(I^{\beta(2-\alpha)} \frac{d^2}{dt^2} I^{(1-\beta)(2-\alpha)} f \right)(t) \text{ for a.e. } t \in I, \quad 1 < \alpha < 2.$$

There are special cases that must be mentioned

When $\beta = 0$ we obtain the Riemann-Liouville derivative:

$$D^{\alpha,0}f(t) = \frac{d^2}{dt^2} I^{(2-\alpha)}f(t).$$

When $\beta = 1$ we obtain the Caputo derivative:

$$D^{\alpha,1}f(t) = I^{(2-\alpha)} \frac{d^2}{dt^2} f(t).$$

Therefore, the initial conditions with Hilfer are given in the following form:

$$I^{(1-\beta)(2-\alpha)}x(0) = x_0, \quad \frac{d}{dt}(I^{(1-\beta)(2-\alpha)}x)(0) = x_1.$$

Theorem 1. (Emmanuele's Fixed Point Theorem) [7]

Let Q be a nonempty, closed, bounded, compact in measure and convex subset of a real Banach space $L_1(0, 1)$ and $T : Q \rightarrow Q$ be β – condensing

i. e. it is continuous and have the property that there is a constant $\xi \in [0, 1)$ such that $\beta(TX) \leq \xi\beta(X)$ for any subset X of Q , where $\beta(X)$ is the De Blas measure of weak noncompactness given by

$$\beta(X) := \lim_{\epsilon \rightarrow 0} \sup_{x \in X} (\sup_D [\int_D |x(t)| dt : D \subset (0, 1), \text{meas } D \leq \epsilon]).$$

Then T has at least one fixed point in Q .

Theorem 2. (Lebesgue's Dominated Convergence Theorem) [10]

Let $\{f_n\}$ be a sequence of real-valued measurable functions defined on a measurable set E . Suppose that the sequence converges pointwise almost everywhere on E to a function f , *i. e.*, $f_n(x) \rightarrow f(x)$ *a. e. on* E .

If there exists an integrable function $g \in L_1(E)$ such that

$$|f_n(x)| \leq g(x)$$

for all n and almost everywhere on E , then f is integrable ($f \in L_1(E)$) and

$$\lim_{n \rightarrow \infty} \int_E f_n dx = \int_E f dx.$$

In the following Lemma, we indicate some important properties of the fractional integral operators

Lemma 1. (Semi-group property) [15]

Let $\alpha, \beta > 0$. If $f \in L_1[0, b]$, then $I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t)$ is satisfied at almost every point $t \in [0, b]$ This property is true in any point if $f \in C[0, b]$ In general, we have

$$(I^\alpha)^n f(t) := I^\alpha I^\alpha \dots I^\alpha f(t) = I^{n\alpha} f(t), \quad a. e. \text{ in } [0, b], n = 1, 2$$

Lemma 2. (Boundedness and continuity of I^α) [8]

Let $\alpha > 0$. Then the following holds

1. $I^\alpha : L_p[0, b] \rightarrow L_p[0, b]$ is bounded linear (hence continuous) operator for every $p \in [1, \infty]$. That is, the fractional integral operator maps $L_p[0, b]$ continuously into itself. In particular, if $0 < \alpha < 1$, then $I^\alpha : L_1[0, b] \rightarrow L_{1/(1-\alpha)-\epsilon}[0, b]$ however small $\epsilon > 0$ is.

2. $I^\alpha : C[0, b] \rightarrow C[0, b]$ is bounded linear (hence continuous) operator.

That is, the fractional integral operator maps $C[0, b]$ continuously into itself. In addition,

$$I^\alpha f(0) := \lim_{t \rightarrow 0} I^\alpha f(t) = 0.$$

Lemma 3. (Preservation of nonnegative nondecreasing functions) [8]

The fractional integral operator maps, nonnegative *a. e.* nondecreasing functions continuously into a function of the same type.

The existence of a unique monotonic solution to a fractional differential

The existence of a unique monotonic solution to a fractional differential equation with multiple non-independent terms within the Banach space of integrable functions. In our proof, we rely mainly on the concept of noncompacting measurement and the fixed-point theory of G. Emanuel the focus here is on proving existence of non-decreasing solution to this class of non-independent polynomial differential equations of the type.

$$\begin{cases} Ax''(t) + BD^{3/2,\beta}x(t) + Cx(t) = f(t, x(\varphi(t))) \text{ a. e. } t \in (0,1) \\ I^{(1-\beta)/2}x(0) = x_0, \quad \frac{d}{dt}[I^{(1-\beta)/2}]x(0) = x_1 \end{cases} \quad (1)$$

Where $0 \leq \beta \leq 1$, $D^{3/2,\beta}$ It is a Hilfer operator of order $3/2$, and $A > 0$, $B, C < 0$.

Definition of the Hilfer operator

$$D^{3/2,\beta}x(t) = I^{\beta/2} \frac{d^2}{dt^2} [I^{(1-\beta)/2}x(t)]$$

Dividing the equation by A (where $A \neq 0$), we obtain:

$$x''(t) + \frac{B}{A}D^{3/2,\beta}x(t) + \frac{C}{A}x(t) = \frac{1}{A}f(t, x(\varphi(t)))$$

We apply the fractional integral I^2 to both sides

$$I^2 x''(t) + \frac{B}{A}I^2 D^{3/2,\beta}x(t) + \frac{C}{A}I^2 x(t) = \frac{1}{A}I^2 f(t, x(\varphi(t)))$$

We calculate the first term

$$I^2 x''(t) = x(t) - x(0) - tx'(0)$$

Using the initial

$$D^{3/2,\beta}x(t) = I^{\beta/2} \frac{d^2}{dt^2} I^{(1-\beta)/2}x(t)$$

And we have

$$I^2 D^{3/2,\beta}x(t) = I^2 \left[I^{\beta/2} \frac{d^2}{dt^2} I^{(1-\beta)/2}x(t) \right] = I^{2+\frac{\beta}{2}} \frac{d^2}{dt^2} I^{2+\frac{\beta}{2}}x(t)$$

To simplify the calculation, let $y(t) = I^{(1-\beta)/2}x(t)$ and $y'(t) = \frac{d}{dt}I^{(1-\beta)/2}x(t)$

Also,

$$y''(t) = \frac{d^2}{dt^2}I^{(1-\beta)/2}x(t)$$

The issue then becomes

$$I^2 D^{3/2,\beta}x(t) = I^{2+\frac{\beta}{2}}y''(t)$$

We use the fundamental property of the fractional integral of the second derivative

$$\begin{aligned} I^{2+\frac{\beta}{2}}y''(t) &= I^{2+\frac{\beta}{2}-2}y(t) - y(0)\frac{t^{2+\frac{\beta}{2}-1}}{\Gamma\left(2+\frac{\beta}{2}\right)} - y'(0)\frac{t^{2+\frac{\beta}{2}-2}}{\Gamma\left(2+\frac{\beta}{2}-1\right)} \\ &= I^{\frac{\beta}{2}}y(t) - y(0)\frac{t^{1+\frac{\beta}{2}}}{\Gamma\left(2+\frac{\beta}{2}\right)} - y'(0)\frac{t^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)} \end{aligned}$$

Since

$$I^{\frac{\beta}{2}}y(t) = I^{\frac{\beta}{2}}I^{(1-\beta)/2}x(t) = I^{\beta/2+(1-\beta)/2}x(t) = I^{1/2}x(t)$$

And

$$I^{\frac{1-\beta}{2}}x(0) = x_0.$$

We knew $y(t) = I^{(1-\beta)/2}x(t)$, then $y(0) = x_0$. Consequently

$$-y(0)\frac{t^{1+\frac{\beta}{2}}}{\Gamma\left(2+\frac{\beta}{2}\right)} = -x_0\frac{t^{1+\frac{\beta}{2}}}{\Gamma\left(2+\frac{\beta}{2}\right)}$$

From second condition then $y'(0) = x_1$, hence

$$-y'(0)\frac{t^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)} = -x_1\frac{t^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)}$$

Therefor

$$I^2 D^{3/2, \beta} x(t) = I^{1/2} x(t) - x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma(2+\frac{\beta}{2})} - x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma(1+\frac{\beta}{2})}.$$

We substitute into all the terms

$$\begin{aligned} x(t) - x_0 - tx_1 + \frac{B}{A} \left[I^{1/2} x(t) - x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma(2+\frac{\beta}{2})} - x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma(1+\frac{\beta}{2})} \right] &+ \frac{C}{A} \int_0^t (t-s) x(s) ds \\ &= \frac{1}{A} \int_0^t (t-s) f(s, x(\varphi(s))) ds \\ x(t) = x_0 + tx_1 - \frac{B}{A} \left[I^{1/2} x(t) + x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma(2+\frac{\beta}{2})} + x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma(1+\frac{\beta}{2})} \right] \\ &- \frac{C}{A} I^2 x(t) + \frac{1}{A} I^2 f(t, x(\varphi(t))) \end{aligned} \quad (2)$$

This is the integral equation equivalent to the differential equation (1).

Assumptions

(1) Let G be a finite subset of the set of real numbers $\mathbb{R}$ and let the function f be defined as follows $f: (0,1) \times G \rightarrow \mathbb{R}$

So that it achieves the following characteristics:

(a) The function $f(t, \cdot)$ is continuous for each $t \in (0,1)$.

(b) The function $f(\cdot, x)$ is measurable for every $x \in G$.

(c) There exist two real functions $a(\cdot), b(\cdot)$ such that:

$$|f(t, x)| \leq a(t) + b(t)|x| \text{ for each } t \in (0,1), x \in G.$$

where $a \in L_1(0,1)$ and b is measurable and bounded, i.e. $|b(t)| \leq m, m \geq 0$.

(d) For every non-decreasing function x transformation $t \rightarrow f(t, x(\varphi(t)))$ is non-decreasing.

(2) $\varphi: (0,1) \rightarrow (0,1)$ is nondecreasing, absolutely continuous and there is a constant $M > 0$ such that $\varphi' \geq M$ a.e on $(0,1)$.

Theorem 3.

Let $0 \leq \beta \leq 1$, and $A > 0, B, C < 0$. Assume that the hypotheses (1-2) are satisfied. If

$$-\frac{|B|}{A\Gamma(3/2)} - \frac{|C|}{2A} + \frac{m}{2AM} < 1.$$

Then the fractional Bagley–Torvik problem (1) admits a unique nonnegative monotonic solution $x \in L_1(0,1)$.

We define the operator $T, T: L_1(0,1) \rightarrow L_1(0,1)$ by

$$(Tx)(t) = x_0 + tx_1 - \frac{B}{A} \left[I^{1/2}x(t) + x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma\left(2+\frac{\beta}{2}\right)} + x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)} \right] - \frac{C}{A} I^2x(t) + \frac{1}{A} I^2f\left(t, x(\varphi(t))\right) \quad (3)$$

And this is the definition of the operator T that we will use.

We determine the set of non-decreasing solutions (we know the set k)

$$k = \{x \in L_1: x(t) \text{ non-decreasing } [0,1]\}$$

In other words, $x \in k \Leftrightarrow t_1 \leq t_2 \Rightarrow x(t_1) \leq x(t_2)$.

Let us take $x \in k$, then we take $0 \leq t_1 \leq t_2 \leq 1$, then

$$(Tx)(t_2) - (Tx)(t_1) = x_1(t_2 - t_1) - \frac{B}{A} \left[I^{1/2}[x(t_2) - x(t_1)] + x_0 \frac{t_2^{1+\frac{\beta}{2}} - t_1^{1+\frac{\beta}{2}}}{\Gamma\left(2+\frac{\beta}{2}\right)} + x_1 \frac{t_2^{\frac{\beta}{2}} - t_1^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)} \right] - \frac{C}{A} [I^2x(t_2) - I^2x(t_1)] + \frac{1}{A} [I^2f(t_2, x(\varphi(t_2))) - I^2f(t_1, x(\varphi(t_1)))]$$

Since $A > 0$, $B, C < 0$. Also, $t \rightarrow f(t, x(\varphi(t)))$ is non-decreasing and φ is nondecreasing. So

$$1 + \frac{\beta}{2} > 0 \Rightarrow t_2^{1+\frac{\beta}{2}} - t_1^{1+\frac{\beta}{2}} \geq 0.$$

We obtain (with the appropriate monotonicity assumption) that the difference is non-negative, then

$$(Tx)(t_2) \geq (Tx)(t_1)$$

Consequently Tx is non-decreasing, then $T(k) \subseteq k$.

Now we prove that $T \in L_1(0,1)$ i.e. $\|Tx\| < \infty$.

$$(Tx)(t) = x_0 + tx_1 - \frac{B}{A} \left[I^{1/2}x(t) + x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma\left(2+\frac{\beta}{2}\right)} + x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)} \right] - \frac{C}{A} I^2x(t) + \frac{1}{A} I^2f\left(t, x(\varphi(t))\right)$$

Then

$$\begin{aligned} \|Tx\| &\leq \int_0^1 |x_0| dt + \int_0^1 |tx_1| dt - \frac{|B|}{A} \left[\frac{\|x\|}{\Gamma(3/2)} + x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma\left(2+\frac{\beta}{2}\right)} + x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)} \right] \\ &\quad - \frac{|C|}{A} \frac{\|x\|}{\Gamma(3)} - \frac{1}{A} \|I^2f(t, x(\varphi(t)))\| \\ &\leq |x_0| + \frac{|x_1|}{2} - \frac{|B|}{A} \left[\frac{\|x\|}{\Gamma(3/2)} + x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma\left(2+\frac{\beta}{2}\right)} + x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)} \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{|C|}{A} \frac{\|x\|}{\Gamma(3)} - \frac{1}{A} \frac{1}{2} \left[\|a\| + \frac{m}{M} \|x\| \right]. \\
& \leq |x_0| + \frac{|x_1|}{2} - \frac{|B|}{A} \left[x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma\left(2+\frac{\beta}{2}\right)} + x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)} \right] - \frac{\|a\|}{2A} \\
& + \left[-\frac{|B|}{A\Gamma(3/2)} - \frac{|C|}{2A} - \frac{m}{2AM} \right] \|x\|
\end{aligned}$$

Since $0 \leq \beta \leq 1$ for $\frac{\beta}{2} \geq 0$, consequently $t^{1+\frac{\beta}{2}}$, $t^{\frac{\beta}{2}}$ integrable on $(0,1)$.

Therefore, $\|Tx\| < \infty$. Then $Tx \in L_1(0,1)$.

Now, we prove that T is continuous

Let $\{x_n\}$ be a sequence in $L_1(0,1)$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$ with respect to the norm $\|\cdot\|$, i.e.

$$\lim_{n \rightarrow \infty} \|x_n - x\| = 0$$

Consequently, there exists a subsequence (denoted again by $\{x_n\}$ for simplicity) that converges pointwise a.e. to x on $(0,1)$

$x_n \rightarrow x$ a.e. on $(0,1)$.

We write Equation (3), and then we replace every x in Equation (3) with x_n

$$\begin{aligned}
(Tx_n)(t) &= x_0 + tx_1 - \frac{B}{A} \left[I^{1/2}x_n(t) + x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma\left(2+\frac{\beta}{2}\right)} + x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)} \right] - \frac{C}{A} I^2x_n(t) \\
&+ \frac{1}{A} I^2f\left(t, x_n(\varphi(t))\right)
\end{aligned}$$

We subtract Tx from Tx_n , we observe that the terms x independent of x_0, x_1 that is, they equal zero

$$Tx_n - Tx = -\frac{B}{A} [I^{1/2}x_n - I^{1/2}x] - \frac{C}{A} [I^2x_n - I^2x] + \frac{1}{A} I^2 \left[f\left(t, x_n(\varphi(t))\right) - f\left(t, x(\varphi(t))\right) \right]$$

We take the standard $\|Tx_n - Tx\|$, using the triangle inequality. We get

$$\begin{aligned}
\|Tx_n - Tx\| &\leq \frac{|B|}{A} \|I^{1/2}(x_n - x)\| - \frac{|C|}{A} \|I^2(x_n - x)\| + \frac{1}{A} \left\| I^2 \left[f\left(t, x_n(\varphi(t))\right) - f\left(t, x(\varphi(t))\right) \right] \right\| \\
&\leq \frac{|B|}{A\Gamma(3/2)} \|x_n - x\| + \frac{|C|}{2A} \|x_n - x\| \\
&+ \frac{1}{2A} \left\| f\left(t, x_n(\varphi(t))\right) - f\left(t, x(\varphi(t))\right) \right\|
\end{aligned}$$

Since φ is nondecreasing and absolutely continuous (Assumption (2)), it preserves sets of measure zero.

Therefore, we have

$$x_n(\varphi(t)) \rightarrow x(\varphi(t)) \text{ a.e. on } (0,1).$$

By Assumption (1-a), the function $f(t, \cdot)$ is continuous with respect to its second argument for each $t \in (0,1)$. Hence, the pointwise convergence yields

$$f(t, x_n(\varphi(t))) \rightarrow f(t, x(\varphi(t))) \quad a. e. on (0,1)$$

Furthermore, by Assumption (1 - c), f satisfies the growth condition

$|f(t, x(t))| \leq a(t) + b(t)|x|$ for each $t \in (0,1)$ and $x \in G$, where $a \in L_1(0,1)$ and b is bounded and measurable. Applying the triangle inequality gives:

$$|f(t, x_n(\varphi(t))) - f(t, x(\varphi(t)))| \leq a(t) + b(t)(|x_n(\varphi(t))| + |x(\varphi(t))|)$$

Since the sequence $\{x_n\}$ is convergent in $L_1(0,1)$, and a subsequence $\{x_{n_k}\}$ such that $|x_{n_k}(t)| \leq h(t)$ a. e. on $(0,1)$. Thus, the sequence of functions $t \rightarrow f(t, x_{n_k}(\varphi(t))) - f(t, x(\varphi(t)))$ is dominated by an integrable function

$$g(t) = 2a(t) + b(t)(h(\varphi(t)) + |x(\varphi(t))|) \in L_1(0,1)$$

By applying Lebesgue's Dominated Convergence Theorem, it follows that

$$\lim_{n \rightarrow \infty} \left\| I^2 \left[f(t, x_n(\varphi(t))) - f(t, x(\varphi(t))) \right] \right\| = 0$$

We conclude that

$$\lim_{n \rightarrow \infty} \|Tx_n - Tx\| = 0$$

This prove that the operator T is sequentially continuous.

If T is continuous over k .

We want to find a small, limited subset to work within, rather than working on the entire set k , therefore, we defined

$$k - R = \{x \in k: \|x\| \leq R\}$$

Where $R > 0$ is a number we choose

Then we want to prove $T(k - R) \subseteq k - R$, that is, if we take any $x \in k - R$ for $Tx \in k - R$ from the definition of $k - R$ we know that $\|x\| \leq R$ if we establish $\|Tx\| \leq R$ it is well known

$$(Tx)(t) = x_0 + tx_1 - \frac{B}{A} \left[I^{1/2}x(t) + x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma(2+\frac{\beta}{2})} + x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma(1+\frac{\beta}{2})} \right] - \frac{C}{A} I^2x(t) + \frac{1}{A} I^2f(t, x(\varphi(t)))$$

We take the norm of both sides using the triangle inequality

$$\begin{aligned} \|Tx\| &= \|x_0\| + \|tx_1\| - \frac{|B|}{A} \left[\|I^{1/2}x\| + \left\| x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma(2+\frac{\beta}{2})} \right\| + \left\| x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma(1+\frac{\beta}{2})} \right\| \right] - \frac{|C|}{A} \|I^2x\| \\ &\quad + \frac{1}{A} \|I^2f(t, x(\varphi(t)))\| \end{aligned}$$

Since $\|x\| \leq R$, then

$$\begin{aligned} \|Tx\| &\leq \|x_0\| + \|tx_1\| - \frac{|B|}{A} \left[\frac{R}{\Gamma(3/2)} + \left\| x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma\left(2+\frac{\beta}{2}\right)} + \left\| x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)} \right\| \right] - \frac{|C|}{A} \frac{R}{\Gamma(3)} \\ &\quad + \frac{1}{2A} [\|a\| + (m/M)R] \\ &\leq \|x_0\| + \|tx_1\| - \frac{|B|}{A} \left[\left\| x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma\left(2+\frac{\beta}{2}\right)} + \left\| x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)} \right\| \right] + \frac{\|a\|}{2|A|} \\ &\quad + \left[-\frac{1}{\Gamma(3/2)} \frac{|B|}{A} - \frac{|C|}{2A} + \frac{1}{2A} (m/M) \right] R \end{aligned}$$

Then

$$\|Tx\| \leq C_0 + C_1 R$$

Where C_0 contains the terms that do not depend on x , and C_1 represents the terms that depend on the norm of x

Now, we have

$$\|Tx\| \leq C_0 + C_1 R$$

It suffices to choose R such that

$$C_0 + C_1 R \leq R \Rightarrow C_0 \leq R - C_1 R$$

Any

$$C_0 \leq (1 - C_1)R$$

If it was $C_1 < 1$ we can choose R such that:

$R \geq C_0/(1 - C_1)$ and then it is realized $\|Tx\| \leq R$ we proved that $x \in k \Rightarrow Tx \in k$ and given that $k - R \subset k$ so, if we take $x \in k - R$ for $Tx \in k$ from the previous calculation, we have $\|Tx\| \leq R$, thus, the two conditions are met $Tx \in k$ and $\|Tx\| \leq R$ consequently, $Tx \in k - R$, if $T(k - R) \subseteq k - R$.

We now have a bounded set of non-decreasing functions $k - R$

$T: k - R \rightarrow k - R$, and T continues.

Consider a non-empty, bounded set. $x \subseteq k - R$, the degree of non-aggregation is represented by the symbol $\beta(x)$.

We use the measure of non-compactness instead of the compactness condition, and the limits whose impact on non-compactness must be studied are $I^{1/2}x$, I^2x , $I^2f\left(t, x(\varphi(t))\right)$ depending on the appropriate property of the measure of non-compactness we are using, we obtain

$$\beta(I^{1/2}x) \leq 1/\Gamma(3/2)\beta(x)$$

then,

$$\beta\left(\left(\frac{B}{A}\right)I^{1/2}x\right) \geq \frac{|B|}{A\Gamma(3/2)}\beta(x)$$

The non-compacting estimate for the I^2x term gives

$$\beta(I^2x) \leq \frac{1}{2}\beta(x)$$

$$\beta(x) \left(\left(\frac{C}{A} \right) I^2x \right) \leq \frac{|C|}{2A}\beta(x)$$

And we have $f(t, x(\varphi(t)))$ we previously assumed that f satisfies the Carathéodory condition *i. e.*

$$\beta(f(x(\varphi))) \leq \frac{m}{M}\beta(x)$$

So,

$$\beta(I^2 [f(t, x(\varphi(t)))]) \leq \frac{m}{2M}\beta(x)$$

Then

$$\beta\left(\left(\frac{1}{A}\right) I^2 [f(t, x(\varphi(t)))]) \leq \frac{m}{2AM}\beta(x)$$

We get

$$\beta(Tx) \leq \left[-\frac{|B|}{A\Gamma(3/2)} - \frac{|C|}{2A} + \frac{m}{2AM} \right] \beta(x)$$

Instead of writing this expression let's put

$$q = \left[-\frac{|B|}{A\Gamma(3/2)} - \frac{|C|}{2A} + \frac{m}{2AM} \right]$$

Then

$$\beta(Tx) \leq q\beta(x)$$

We require $q < 1$ because we want to reduce T non-compacting and therefore

$$\beta(Tx) \leq q\beta(x) < \beta(x)$$

It is well known that $\beta(x) > 0$. If T is a reducing effect to non-compacting

Given that the operator T is continuous, satisfies the condition $T(k - R) \subseteq k - R$ and is a non-compactness reducing operator on $k - R$, it follows from Emmanul's fixed point theorem that there exists a function $x \in k -$

R satisfying $Tx = x$. By the definition of the operator T , this fixed point satisfies the integral equation associated with the Bagley-torvik problem featuring the Hilfer derivative and the prescribed initial conditions

Since $x \in k - R$ holds x is non-decreasing on $[0,1]$. Consequently, the Bagley-Torvik problem involving the Hilfer operator possesses a non-decreasing solution.

Since a decreasing solution has been found, we now wish to prove that this solution is unique.

Suppose there are two non-decreasing solution to the problem, and let us call them x_1 and x_2 . since the two solutions are non-decreasing, they belong to $k - R$, and therefore $x_1, x_2 \in k - R$. Since every solution to the differential problem satisfies the equivalent integral equation, $x_1^* = Tx_1^*$ and $x_2^* = Tx_2^*$,

let us propose

$$x_1^* - x_2^* = Tx_1^* - Tx_2^*$$

We have

$$\begin{aligned} Tx_1^* - Tx_2^* = & -\frac{B}{A} \left[I^{1/2}(x_1^* - x_2^*) + x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma\left(2+\frac{\beta}{2}\right)} + x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)} \right] \\ & -\frac{C}{A} I^2(x_1^* - x_2^*) + \frac{1}{A} I^2 \left[f\left(t, x_1^*(\varphi(t))\right) - f\left(t, x_2^*(\varphi(t))\right) \right] \end{aligned}$$

By taking the norm of both sides and using the triangle inequality

$$\begin{aligned} \|x_1^* - x_2^*\| \leq & -\frac{|B|}{A} \left[\|I^{1/2}(x_1^* - x_2^*)\| + \left\| x_0 \frac{t^{1+\frac{\beta}{2}}}{\Gamma\left(2+\frac{\beta}{2}\right)} \right\| + \left\| x_1 \frac{t^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)} \right\| \right] - \frac{|C|}{A} \|I^2(x_1^* - x_2^*)\| \\ & + \frac{1}{A} \left\| I^2 \left[f\left(t, x_1^*(\varphi(t))\right) - f\left(t, x_2^*(\varphi(t))\right) \right] \right\| \\ \leq & -\frac{|B|}{A\Gamma(3/2)} \|x_1^* - x_2^*\| - \frac{|C|}{2A} \|x_1^* - x_2^*\| + \frac{m}{2AM} \|x_1^* - x_2^*\| \end{aligned}$$

By factoring out $\|x_1^* - x_2^*\|$

$$\|x_1^* - x_2^*\| \leq \left[-\frac{|B|}{A\Gamma(3/2)} - \frac{|C|}{2A} + \frac{m}{2AM} \right] \|x_1^* - x_2^*\|$$

We already knew q

$$q = \left[-\frac{|B|}{A\Gamma(3/2)} - \frac{|C|}{2A} + \frac{m}{2AM} \right]$$

Therefore

$$\|x_1^* - x_2^*\| \leq q \|x_1^* - x_2^*\|$$

So,

$$\|x_1^* - x_2^*\| - q \|x_1^* - x_2^*\| \leq 0$$

By factoring out $\|x_1^* - x_2^*\|$

$$(1 - q) \|x_1^* - x_2^*\| \leq 0.$$

Will use the condition that $q < 1$, this leads to $1 - q > 0$, At the same time $\|x_1^* - x_2^*\| \geq 0$.

Therefore, the previous inequality cannot be fulfilled unless $\|x_1^* - x_2^*\| = 0$.

This means $x_1^*(t) = x_2^*(t) \quad \forall t \in (0,1)$, then $x_1^* = x_2^*$.

Consequently, we cannot have two different solutions.

The Bagley-Torvik problem with the Hilfer derivative posses a non-decreasing solution, and this solution is unique in $L_1(0,1)$.

Conclusion

In this paper, we have investigated the existence and uniqueness of monotone solutions for a class of fractional differential equations in the Banach space L_1 . By formulating the problem into an equivalent integral equation and constructing an appropriate operator, we established the existence of a unique monotone and non-decreasing solution under suitable conditions. The primary theoretical framework relied on the theory of measures of noncompactness combined with an appropriate fixed-point theorem. The results obtained in this study contribute to the advancement of fractional calculus, and the proposed approach can be further extended to analyze more complex systems of fractional differential equations in future research.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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