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On TNC-Rings

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Abstract:

This paper is a continuation of study rings relative to clean ring, where we study the concepts of 2-nil clean rings TN-Clear Ring (TNC-ring) of a noncommutative ring, which introduced by Chen and Sheibani. These rings of ours naturally generalize the nil -clean rings. we show that, if $\boldsymbol{a \epsilon C}$ is TNC - element, then $\boldsymbol{a(a-1)(a-2)}$ is nilpotent and $\boldsymbol{a=k+b}$, where $\boldsymbol{k \in C}$ and $\boldsymbol{b \in N(C)}$, such that; $\boldsymbol{k(k-1)(k-2)=0}$, when every two pair of idempotent elements are orthogonal, and we determine condition under which the TNC-ring is strongly $\boldsymbol{\pi}$-Regular and we show that, if $\boldsymbol{C}$ is commutative local ring and $\boldsymbol{L}$ is maximal ideal. Then $\boldsymbol{C}$ is TNC-ring if $\boldsymbol{C/L \cong Z_3}$.

Keywords: Clean Ring, Local Ring, Nil-Clean Ring, TNC-Ring.

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Introduction

When two idempotents and a nilpotent element are added together, Chen and Sheibani describe an element of a ring C as a 2-nil-clean element (TN Clean -element) [1]. A TNC -ring is defined to be one in which every element is TNC, Z_{16} (ring of integers modulo16) is TNC-ring. The purpose of this paper is to study a new kind of rings which generalize of nil- clean which introduce by many mathematicians [2,3,4,6], Some mathematicians are interested in the topics of clean ring [6,7,8], A ring is considered to be strongly π −regalar if for every element $a \epsilon C$ there exists $m \geq 1$ and $y \epsilon C$ such that $a^m = a^{m+1}y$[9].

In this paper, we begin by establishing, the basic properties of TNC- ring, Finally, we fined the relationship between TNC- ring and clean - ring, stronglyπ - regular ring and local ring. The notational conventions are that all rings will be associative with identity.

The Jacobson radical, the set of all nilpotent element in C, the set of all unites elements and the set of all idempotent elements in C are denoted by $J(C), N(C), U(C)$ and $Id(C)$.

The following definition was defined in [6,7,8]

Definition 1:

A ring C is called clean if, for each a in R, there exists $u \epsilon U(C)$ and $e \epsilon Id(C)$ such that, $a = u + e$, furthermore, if $eu = ue$the clean ring C is said to be strongly clean.

On the other side in [2,3,4,5] were introduced the following concept:

Definition 2:

A ring C is called nil - clean if, for every a in R, there are $b \epsilon N(C)$ and $e \epsilon Id(C)$, with $a = e + b$. If $eb = be$ is satisfied, then we say that C is strongly nil - clean ring.

Chen and Shebani in [1] give the definition of strongly 2-nil clean rings (STNC - rings)

Definition 3:

A ring C is said to be strongly 2-nil clean (STNC -ring), If for every a in C, there are $b \epsilon N(C)$ and $e_1, e_2 \epsilon Id(C)$, with $a = e_1 + e_2 + b$, that commutes with others.

For example, the ring of integers modulo 16,(Z_{16}) is STNC-ring.

Definition 4:

A ring C is called abelian if every idempotent element in ring C is central [10].

Material and methods

The object of this section is to investigate certain basic properties of TNC- rings

Proposition 1:

Let D be a nil ideal of a ring C. Then C is TNC - ring if and only if, C/D is TNC - ring.

Proof: suppose that C/D is TNC-ring and let $a \epsilon C$. Than, $a + D \epsilon C/D$,then there exists $e_1, e_2 \epsilon Id(C)$ and $b \epsilon N(C)$ such that $a + D = (e_1 + D) + (e_2 + D) + (b + D)$. It follows $a + D = (e_1 + e_2 + b) + D$, This implies $a - (e_1 + e_2 + b) \epsilon D$. Since D is a nil ideal of C, therefore idempotents lift modulo D by [2, Proposition 3.15]. Also, $b \epsilon N(C)$, It follows that $a - (e_1 + e_2)$ is nilpotent modulo D is nil and D is nil, then $a - (e_1 + e_2 + b) = b_1$ where b1 is nilpotent. Then we get $a = e_1 + e_2 + b + b_1$. Hence $a = e_1 + e_2 + b_2$. Therefore D is TNC-ring.

Converse is clear from TNC-rings are closed under homomorphic images.

Proposition 2:

Let C be a ring with every pair of idempotents are orthogonal Then an element x in C is TNC-element if and only if, $(1 - x)$ is TNC element.

Proof: Assume that x is 2- TNC-element, then $x = e_1 + e_2 + b$, where $e_1, e_2 \epsilon Id(C)$ and $b \epsilon N(C)$,Then $1 - x = 1 - (e_1 + e_2) + (-b)$, since $e_1, e_2$are orthogonal. Then $(e_1 + e_2)^2 = e_1 + e_2$.

Thus $(1 - (e_1 + e_2))$ is idempotent and $b \epsilon N(C)$, that is $b^n = 0$, nϵZ+, then $(-b)^n = 0$,when n is even positive integer. Therefore$(1 - x)$ is 2-TNC-element.

Conversely: Suppose that $(1 - x)$ is 2-nil clean element, Then $1 - x = e_1 + e_2 + b$ where $e_1, e_2 \epsilon Id(c)$ and $b \epsilon N(C)$

Hence, $-x = (e_1 + e_2) - 1 + b$. Thus,$x = 1 - (e_1 + e_2) - b$. Then $x = (1 - + (e_1 + e_2)) + (-b)$.

Now, Since $(e_1 + e_2) \epsilon Id(C)$, then$\left(1 + (e_1 + e_2)\right) \epsilon$ Id (C) and $(-b) \epsilon N(C)$.

Therefore x is 2-nil clean.

Proposition 3:

Let C be an abelian ring, with every two pair of idempotents are orthogonal. Then any element x in C is TNC – element, if and only if x^n is TNC - element, for some positive integer n.

Proof: we prove by mathematical induction

Let $x$be 2-nil clean, that is $x = e_1 + e_2 + b, where\ e_1, e_2 \epsilon Id(C)\ and\ b \epsilon N(C)$, and $x^2 = (e_1 + e_2 + b)^2 = (e_1 + e_2)^2 + 2b\,(e_1 + e_2) + b^2 = e_1 + e_2 + b', e_1 + e_2 \epsilon Id(C)$ and $b' = 2b(e_1 + e_2) + b^2 \epsilon N(C)$

Now, assume that for $n = m - 1$,the assumption is true, that; $x^{m-1} = (e_1 + e_2 + b)^{m-1} = \sum_{k=0}^{m-1} \binom{m}{k} (e_1 + e_2)^k b^{m-k} = (e_1 + e_2)^{m-1} + \frac{(m-1)}{1!}(e_1 + e_2)^{m-2} + \frac{(m-1)(m-2)}{2!}(e_1 + e_2)^{m-3} b^2 + \frac{(m-1)(m-2)(m-3)}{3!}(e_1 + e_2)^{m-4} b^3 + \cdots \ldots . + b^{m-1}$

We must prove the assumption is true when $n = m$,

$x^m = x\,x^{m-1} = (e_1+e_2+b)\Big[(e_1+e_2)^{m-1} + \frac{(m-1)}{1!}(e_1+e_2)^{m-2}\,b + \frac{(m-1)(m-2)}{2!}(e_1+e_2)^{m-3}b^2 + \frac{(m-1)(m-2)(m-3)}{3!}(e_1+e_2)^{m-4}b^3 + \cdots \ldots . + b^{m-1}\Big]$

$= (e_1+e_2)^m + b[\,(e_1+e_2)^{m-1} + \frac{m(m-1)}{2!}(e_1+e_2)^{m-2}\,b + \frac{m(m-1)(m-2)}{3!}(e_1+e_2)^{m-3}b^2 \ldots \ldots . + b^m]$

Suppose that, $L = b[\,(e_1+e_2)^{m-1} + \frac{m(m-1)}{2!}(e_1+e_2)^{m-2}\,b + \frac{m(m-1)(m-2)}{3!}(e_1+e_2)^{m-3}b^2 + \cdots + b^m]$, Hence $L\,\epsilon N(C) = \sum_{K=0}^{M}\binom{m}{k} e_1^k e_2^{m-k} + L = [{e_1}^m + m{e_1}^{m-1}e_2 + \frac{m(m-1)}{2!}{e_1}^{m-2}e_2^2 + \frac{m(m-1)(m-2)}{3!}{e_1}^{m-3}e_2^3 + \cdots + e_2^m] + L = {e_1}^m + e_2^m + L$

Thus, $x^m = {e_1}^m + e_2^m + L$, Therefore x^m is 2-nil clean

Conversely, suppose that x^n be TNC-element, that is $x^n = e_1 + e_2 + b$, $e_1 + e_2 \epsilon Id(c)$, $b \epsilon N(c)$

$x^n - (e_1+e_2) = b$

$x^n - (e_1+e_2) = (x-(e_1+e_2))[(e_1+e_2) + (e_1+e_2)x + (e_1+e_2)x^2 + \cdots \ldots . . + x^{n-1}]$

Since $x^n - (e_1+e_2)$ is nilpotent. Then $(x-(e_1+e_2))[\,(e_1+e_2) + (e_1+e_2)x + (e_1+e_2)x^2 + \cdots + x^{n-1}]\epsilon N(C)$, Hence $x-(e_1+e_2)\epsilon\, N(C)$, if follows to

$x-(e_1+e_2) = b_1$ So, $x = (e_1+e_2) + b_1$ and therefore x is TNC-element

Proposition 4:

Let C be a belianring with $2\epsilon N(C)$ and let a be TNC - element in C.Then $a^2 - a$ is nilpotent element.

Proof: Let $a\epsilon C$ be 2-nil clean element, that there exists, $e_1, e_2 \epsilon\ Id(C), b\epsilon N(C)$, Such that $a = e_1 + e_2 + b$

$a^2 = (e_1+e_2+b)^2 = (e_1+e_2)^2 + 2b(e_1+e_2) + b^2 = {e_1}^2 + 2e_1e_2 + e_2^2 + 2b(e_1+e_2) + b^2 = e_1 + e_2 + 2e_1e_2 + 2b(e_1+e_2) + b^2 (e_1+e_2)\epsilon Id(C), 2e_1e_2 + 2b(e_1+e_2) + b^2 \epsilon N(C)$

$a^2 - a = (e_1+e_2) + 2e_1e_2 + 2b(e_1+e_2) + b^2 - \big((e_1+e_2)+b\big) = (e_1+e_2) + 2e_1e_2 + 2b(e_1+e_2) + b^2 - \big((e_1+e_2)-b\big)$. This implies that $2e_1e_2 + 2b(e_1+e_2) + b^2 - b\ \epsilon\ N(C)$.

Therefore, $(a^2 - a)$ is nilpotent.

Proposition 5:

Let C be TNC – ring, with every pair of two idempotent elements are orthogonal, then $J(C)$ is nil ideal and $2\epsilon\ J(C)$.

Proof: Let $x\ \epsilon\ J(C), then\ x\epsilon\ C$, by proposition(4), $(x^2 - x)\ \epsilon\ N(C)$ and hence $(x - x^2)\epsilon N(C)$.

Assume that; $w = (x - x^2)\epsilon N(C)$, implies that $w = x(1-x)\ and\ x = w(1-x)^{-1}\epsilon\ N(C)$, Thus J(C) is nil ideal. Now, let $a\ \epsilon\ C$. Then, there exists $e_1, e_2\ \epsilon Id(C)$ and $b\epsilon N(C)$, such that: $a = e_1 + e_2 + b$ Assume that $2 = e_1 + e_2 + b$, since e_1,e_2 are orthogonal, then $e_1 + e_2 = e_3\epsilon\ Id(C)$,That is $2 = e_3 + b$, it follows that $1 + 1 = e_3 + b$, hence $1 - e_3 = b - 1$.Since $(b-1)\ \epsilon\ U(C)$, let $1 - e_3 = u$, then $1 - e_3 = 1$, hence $e_3 = 0$. Thus, $2 = 0 + b$ and 2 is nilpotent. Therefore $2\ \epsilon\ J(C)$

Proposition 6:

Let Cbe a belian ring with every pair of idempotent elements are orthogonal and l, k are TNC-elements in C. Then $l + k$ is TNC -element.

Proof: Let l and k are TNC- elements. Then there exists $e_1, e_2, e_3, e_4\epsilon\ Id(C), b_1, b_2\epsilon N(C)$ such that:

$l = e_1 + e_2 + b_1\ and\ k = e_3 + e_4 + b_2$ then, $l + k = (e_1 + e_2 + b_1) + (e_3 + e_4 + b_2) = (e_1 + e_2) + (e_3 + e_4) + (b_1 + b_2)$. Since every pair of idempotent elements are orthogonal, then:

$(e_1+e_2)^2 = (e_1+e_2) = e_5\epsilon\ Id(C)$ and $(e_3+e_4)^2 = (e_3+e_4) = e_6\epsilon\ Id(C)$, also we have $b_1 + b_2 = b_3\epsilon N(C)$

Hence, $l + k = e_5 + e_6 + b_3$.

Therefore, $l + k$ is TNC–element.

Proposition 7:

Let C be an abelian ring, and $a\ \epsilon\ C$ be TNC- element. Then $a(a-1)(a-2)$ is nilpotent element.

Proof: Let $a\epsilon\ C$, since a is TNC-element, then there exists $e_1, e_2\epsilon Id(C), b\epsilon\ N(C)$ such that $a = e_1 + e_2 + b$

Next, we get that.

$a(a-1)(a-2) = (e_1+e_2+b)\ \ (e_1+e_2+b-1)(e_1+e_2+b-2)\ \ =[\,e_1(e_1+e_2+b-1)\ +\ e_2\,(e_1+e_2+b-1)+\ b(e_1+e_2+b-1)\,](e_1+e_2+b-2\ =[\,{e_1}^2\ +\ e_1e_2+e_1b-e_1+e_2e_1+e_2^2+e_2b-e_2+be_1+be_2+b^2-b](\,e_1+e_2+b-2)=[\,e_1+2e_1b-e_1+e_2+2e_2b-e_2+b^2-b](e_1+e_2+b-2)=(2e_1b+2e_2b+b^2-b)(e_1+e_2+b-2)=2{e_1}^2b+\ 2e_1e_2b+\ 2e_1b^2-4e_1b\ +2e_1e_2b\ +2e_2^2b\ +2e_2b^2-4e_2b\ +e_1b^2+e_2b^2+b^3-2b^2-e_1b\ -e_2b-b^2+2b\ \ =b^3\ +\ 3e_1b^2+3e_2b^2\ -3b^2-3e_1b\ -3e_2b\ +2b\ =b(b^2\ -3b\ +2)+\ 3e_1b(b-1)+\ 3e_2b(b-1)=b(b-2)(b-1)+b(b-1)(3e_1+3e_2)=b(b-1)[\,(b-2)(\,3e_1+3e_2)\,]\,\epsilon N(C)$

Proposition 8:

Let C be an abelian ring with every two pair of idempotent elements are orthogonal, and $a\epsilon C$ is TNC - element. Then $a=\ k\ +\ b$,where $k\epsilon C\ and\ b\epsilon N(C)$,such that $k(k-1)(k-2)\ =0$.

Proof: Let $a=\ e_1+e_2+b, where\ e_1, e_2\epsilon Id(C), b\epsilon\ N(C)$

Now, take; $k=e_1+e_2$, then we have:

$k(k-1)(k-2)\ =\ (e_1+e_2)\,[\,(e_1-1)+e_2][\,(e_1-2)+e_2]\ =\ (e_1+e_2)\,(e_1-1\ +e_2)\,(e_1-2\ +e_2)\ \ \ = (e_1+e_2)\,({e_1}^2\ -2e_1+e_1e_2-e_1+2-e_2\ +e_2e_1-2e_2\ +{e_2}^2)$

$=(e_1+e_2)\,(\,-2e_1\ +2\ -2e_2)=\ -2{e_1}^2+\ 2e_1\ -2e_1e_2-2e_2e_1\ +2e_2-2{e_2}^2=0$

Therefore $k(k-1)(k-2)=0$

THE RELEVANCE BETWEEN TNC -RING AND OTHER RINGS

In this section we construct the relation between TNC - ring and clean ring, local ring and strongly π-regular ring.

Proposition 9:

Let C be an abelian TNC - ring with $2\epsilon N(C)$. Then C is clean.

Proof: suppose that C is TNC -ring and $a\ \epsilon\ C$, then there exists$e_1, e_2\epsilon\ Id(C)\ and\ b\epsilon\ N(C)$, such that: $a-1\ = e_1+e_2+b$. Since, $(e_1+e_2)^2=e_1+e_2+2e_1e_2$, Hence, $a-1\ =\ e_1+e_2+2e_1e_2\ +b, 2e_1e_2\ +b\epsilon N(C)$

$since\ (1+2e_1e_2+b)\in U(C)$.Then $a=(e_1+e_2)+(1+2e_1e_2\ +b\ )$ is clean element.

Proposition 10:

Let C be a commutative local ring and L be a maximal ideal. Then C is TNC - ring, if $C/L\cong Z_3$.

Proof: Since C is local ring, then $J(C)=M$ and every element in $J(C)$ is nilpotent.

Then every element in M is nilpotent, If $a\ \epsilon\ C, then\ a+\ M\ =\ x+\ M$ for some $x\in\{0,1,2\}$.

So, $a=1+1+b$, where b is nilpotent, thus a is TNC-element and therefore,C is TNC-ring.

Proposition 11:

Every STNC - ring is strongly π-regular.

Proof: Since C is STNC- ring, that is C is $(3-1)$-nil clean ring. Now consider 3 is nilpotent element. Then $(3-1)!$ Is invertible, let $a\epsilon C$,Since $a(a-1)(a-2)$ is nilpotent from proposition (7). $since\ (1+2e_1e_2+b)\in U(C)$.Then $a=(e_1+e_2)+(1+2e_1e_2\ +b\ )$ is clean element.

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So, $a=1+1+b$, where b is nilpotent, thus a is TNC-element and therefore,C is TNC-ring.

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Proposition 13:

Every TNC-ring is nil-clean ring, if every pair of two idempotent elements are orthogonal.

Proof: let$C$ be TNC-ring and let $a\,\epsilon C$ with$e_1, e_2\,\epsilon\, Id(C)$ and $b\epsilon\, N(C)$ such that $a=e_1+e_2+b$.Sincee_1, e_2 are orthogonal, then $(e_1+e_2)=e_3\epsilon Id(C)$,It follows that $a=e_3+b$ is nil-clean element and therefore, C is nil-clean ring.

Conclusion

From the study characterization and properties of TNC-ring, we obtain the following results:

- when D be a nil ideal of a ring C. Then C is TNC - ring if and only if, C/D is TNC - ring.
- when C be a belianring with $2\in N(C)$ and let a be TNC- element in C.Then a^2-a is nilpotent element
- when C be TNC – ring, with every pair of two idempotent elements are orthogonal, then J(C) is nil ideal and $2\in J(C)$..
- when C be a belian ring with every pair of idempotent elements are orthogonal and l,k are TNC-elements in C. Then $l+k$ is TNC -element.
- when C be an abelian ring, and $a\epsilon\ C$ be TNC-element. Then$a(a-1)(a-2)$ is nilpotent element.
- when C be an abelian TNC -ring with $2\in N(C)$. Then C is clean.
- when C be a commutative local ring and L be a maximal ideal. Then C is TNC-ring, if $C/L\ \cong\ Z_3$.
- Every STNC - ring is strongly π-regular.

We recommend as further studies, to study 3-nil clean and the relation between 2-nil clean. To study p-nil clean ring and give further properties of such rings.

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