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Abaoub -Shkheam Transform Techniques to Solve Volterra Integral and Volterra Integro-Differential Equations

Nagah A. Elbhilil [1 *], Muna I. Bnis [2], Aml A. Altirban [3]

[1, 2, 3] Department of Mathematics, Faculty of Science, Tripoli University, Tripoli, Libya

[*]*Corresponding author:* nagahabdow@yahoo.com

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Abstract:

In this research, wide-field integral equations were studied due to their physical, engineering, medical, and other applications. The Abaoub-Shkeam transform was used, which is a mathematical tool that showed results in partial differential and integral equations. It was recently developed to obtain the analytical to the solution for the linear Volterra integral equation of the first type and Also the Volterra equation of the second type and the Volterra differential-integral equation. For this, we assume that the transform kernel is a convolution kernel.

Some Applications were shown to demonstrate the efficiency and precision of the Abaoub-Shkheam transform method for resolving types of Volterra integral equations, and Bessel function's Abaoub-Shkheam transform was inferred.

Keywords: Volterra Integral Equations, Abaoub –Shkheam Transform, Inverse Abaoub-Shkheam Transform, Bessel's Functions.

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تقنيات تحويل عبعوب-شخيم لحل معادلات فولتيرا التكاملية و فولتيرا التفاضلية التكاملية

نجاح عبدالمجيد البهيليل[1*]، منى إبراهيم بنيس[2]، أمل عبدالله طربان[3]

[1،2،3] قسم الرياضيات، كليةالعلوم ، جامعة طرابلس، طرابلس، ليبيا

الملخص

في هذا البحث، تمت دراسة المعادلات التكاملية ذات المجال الواسع بسبب تطبيقاتها الفيزيائية والهندسية والطبية وغيرها و تم استخدام تحويل عبعوب-شخيم الذي يعد كأداة رياضية أظهرت نتائج في المعادلات التفاضلية الجزئية والتكاملية وتم تطويره مؤخرًا للحصول على الحل التحليلي لمعادلة فولتيرا التكاملية الخطية من النوع الأول و أيضا معادلة فولتيرا من النوع الثاني و معادلة فولتيرا التفاضلية التكاملية. لهذا، نفترض أن نواة التحويل هي نواة الالتفاف.

وتم عرض التطبيقات لإثبات كفاءة ودقة تقنية تحويل عبعوب - شخيم لحل أنواع معينة من معادلات فولتيرا التكاملية ، و تم استنتاج تحويل عبعوب –شخيم لدالة بيسل.

الكلمات المفتاحية: معادلة فولتيرا التكاملية، تحويل عبعوب - شخيم، تحويل عبعوب – شخيم العكسي، دوال بيسل.

Introduction

Problems in engineering, physics, and applied disciplines represent linear or nonlinear integral equations. There are several ways to accurately solve these equations [1]. Sharma et al. [2-3] studied different integral transformations (Kamal transform, Mahjoub transform, Muhannad transform, Zaki transform) to solve the Aggarwal discussed with other scholars [4-5] the solution of Volterra's linear integral differential equations of the second type using different integral transform (perfect transformation, Mahjoub transform, Abudha transform, Sadiq transform). In [6] Adomian introduced and developed the so-called decomposition method for solving differential equations, integral and partial differential equations.

Many researchers were interested in studying the integral transformation, including a new transformation, the Job-Shechem transformation, and its application[7]. We found that partial differential equations were studied[8]. By [9] the partial linear integral differential equations were solved, Abaob et al. [10] The field decomposition

method was applied to the Volterra integral and the Type II equation. In [11] Abaub found the exact solution of the first-kind linear integral Volterra differential equation using the Abaub-Schichem transformation.

The article is divided into parts: We introduced the concept of the Abaoub -Shkheam Transform, found the formula for the Abaoub -Shkheam transform of Bessel functions,and we deduced the Abaoub -Shkheam Transform for the Volterra integral equation of the first and the second type, as well as the Abaoub -Shkheam Transform for the Volterra integro-differential equation of the second type, and Application of linear Volterra equation in Medical science. finally presenting the results obtained from the study.

Research problem

Integral equations of convolutional type are used Describe many problems in engineering, medicine, and most applied sciences. Solve these equations Using the conversion Abaoub – Shkheam.

Where conversion to solve the Volterra integral equation was discussed to solve this type of equation easily.

In this paper Abaoub-Shkeam integral transform simply converts Ab-Sh T, and we will refer to the Volterra integral equation by VIE.

The Mathematical model

The Ab-Sh T was used mathematical model to solve types of VIE, which are usually solved using the Laplace transform.

Basic concepts

Abaoub-Shkheam transform

Let $f(vt)$ is a piecewise continuous function is defined by Ab-Shkheam T as follows:

$$\mathrm{T}(v,s) = \int_0^\infty e^{\frac{-t}{s}} f(vt)dt = Q\{f(t)\} \qquad ;t \geq 0$$

That integration exists for some s, where $s \in (-t_1, t_2)$.

The inverse Ab-Sh T is:

$$f(t) = Q^{-1}[\mathrm{T}(v,s)] = Q^{-1}\left[\int_0^\infty e^{\frac{-t}{s}} f(vt)dt\right] \qquad ;t \geq 0$$

The model is as shown in the following Figure 1 shows the relationship between Q and Q^{-1}

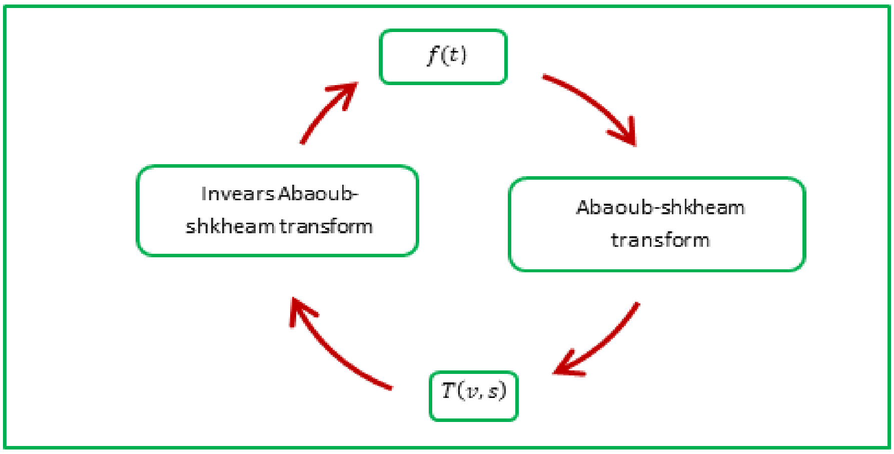

Figure 1 Element Models of Abaoub-Shkheam transform

Table 1 shows the transform of Ab-Sh for some functions:

S.N	$f(t)$	$Q\{f(t)\} = T(v,s)$
1	$t^n \quad ; s > 0, n \in \mathbb{N}$	$n!\ v^n\ s^{n+1}$
2	e^{at}	$\frac{s}{1 - avs}$
3	$cos\ at$	$\frac{s}{1 + a^2v^2s^2}$
4	$sin\ at$	$\frac{avs^2}{1 + a^2v^2s^2}$
5	$cosh\ at$	$\frac{s}{1 - a^2v^2s^2}$
6	$sinh\ at$	$\frac{avs^2}{1 - a^2v^2s^2}$
7	$f^{(n)}(t)$	$\frac{T(v,s)}{v^n s^n} - \frac{1}{v} \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{(v\ s)^{n-k-1}}$

Details of the integral equations and the VIE can be found in [16], and the basics and properties of the Ab-Sh conversion can be introduced [7], but they were not mentioned, In this paper to avoid repetition.

Results and discussion

1. Ab -Sh T of Bessel's Functions

The Bessel function is used to solve many physical and engineering problems that are represented as a partial differential equation or an integral equation.

Laplace transform of Bessel's function [14]of $J_0(t)$and $J_1(t)$:

$$\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{1+s^2}}$$

$$\mathcal{L}\{J_1(t)\} = 1 - \frac{1}{\sqrt{1+s^2}}$$

As for using Ab -Sh T of Bessel's functions

$$Q\{J_0(t)\} = \frac{s}{\sqrt{1+v^2s^2}}$$

$$Q\{J_1(t)\} = \frac{1}{v}\left[1 - \frac{1}{\sqrt{1+v^2s^2}}\right]$$

2. Ab -Sh T For Linear VIE Of First type:

In this work, we will assume that the kernel of equation (1) is a convolutional type kernel that can be expressed as the difference $(x - t)$. Therefore, the general formula [15] for the VIE of the first type (1) is expressed as follows:

$$f(x) = \int_0^x k(x,t)u(t)dt = \int_0^x k(x-t)u(t)dt \qquad (1)$$

Applying the Ab -Sh T to both sides of equation (1). After that, we use the convolution theorem ,

we get:

$$Q\{f(x)\} = v\ Q\{k(x)\}\ Q\ \{u(t)\} \qquad (2)$$

Operating inverse Ab -Sh T on both sides of equation (2), we get:

$$u(x) = Q^{-1}\left\{\frac{Q\{f(x)\}}{v\, Q\{\, k(x)\}}\right\}$$

which is result the solution of equation (1)

3. Ab -Sh T For Linear VIE Of Second Type:

The general form [15] of linera VIE of Second Type is given by

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt \qquad (3)$$

In this research, the kernel of equation (3) is a convolutional kernel. Therefore, equation (3) takes the following form

$$u(x) = f(x) + \lambda \int_0^x k(x-t)u(t)dt \qquad (4)$$

Operating Ab -Sh T on equation (4), and Use of convolution theorem in equation (4) gives

$$Q\{u(x)\} = \frac{Q\{f(x)\}}{1-\lambda v\, Q\{k(x)\}} \qquad (5)$$

After applying the inverse Ab -Sh T to equation (5), the following result is obtained, which represents a solution to equation (3):

4. Ab -Sh T For Linear VI -Differential Of Second Type:

The general form of the integral differential [16] equation of Volterra of the second type

$$u^{(n)}(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt \qquad (6)$$

where $u(x)$ is the solution of the equation; It is very necessary to determine the initial conditions $u(0), u'(0), \dots, u^{(n-1)}(0)$, VI- differential equations can be solved when the initial value problem is in the integral equation.

Use the Ab-Sh T to solve equation (6). The kernel can be expressed as the difference $(x-t)$ in equation (6). Using the convolution theorem for the Ab-Sh T, we get

$$\frac{Q\{u(x)\}}{v^n s^n} - \frac{1}{v}\sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{(v\ s)^{n-k-1}} = Q\{f(x)\} + \lambda v Q\{k(x)\}Q\{u(x)\} \qquad (7)$$

Operating inverse Aba-Sh T on both sides of (7), we have result

$$u(x) = Q^{-1}\left\{\left\{\frac{v^n s^n}{1-\lambda v^{n+1}s^n\, Q\{k(x)\}}\right\}\left\{\frac{1}{v}\sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{(v\ s)^{n-k-1}} + Q\{f(x)\}\right\}\right\}$$

is solution of equation (7).

The following Figure 2 summarizes this paper

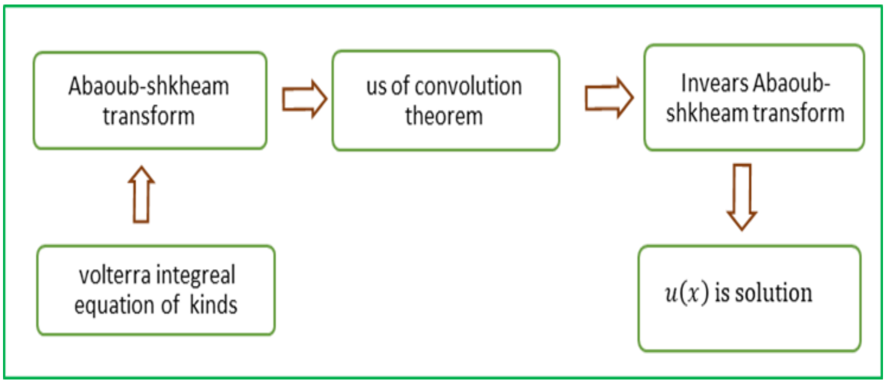

Figure 2 Models of Abaoub-Shkheam transform solution of Volterra

Applications of Ab-Sh T to types VIE :

a) We assume that the linear VIE of the first type is

$$x = \int_0^x e^{(x-t)}u(t)dt$$

solution: Applying Ab -Sh T to both sides and using convolution theorem and simplify, we have

$$Q[u(x)] = s - vs^2$$

Operating inverse Ab -Sh T on both sides, we have

$$u(x) = Q^{-1}[s - vs^2]$$
$$= 1 - x$$

b) We assume that the linear VIE of the first type is

$$\cos x - J_0(x) = -\int_0^x J_1(x-t)u(t)dt$$

solution: Applying Ab –Sh T to both sides and using convolution theorem and simplify, we have

$$Q[u(x)] = \frac{s}{\sqrt{1+v^2s^2}}$$

Operating inverse Ab -Sh T on both sides, we have

$$u(x) = Q^{-1}\left[\frac{s}{\sqrt{1+v^2s^2}}\right]$$
$$= J_0(x)$$

c) We assume that the linear VIE of the second type is

$$u(x) = x + \int_0^x (t-x)u(t)dt$$

solution: Applying Ab -Sh T to both sides and using convolution theorem and simplify, we have

$$Q[u(x)] = \frac{vs^2}{1+v^2s^2}$$

Operating inverse Ab -Sh T on both sides, we have

$$u(x) = Q^{-1}\left[\frac{vs^2}{1+v^2s^2}\right]$$
$$= \sin x$$

d) We assume that the linear VIE of the second type is

$$u(x) - \int_0^x u(t)sin(x-t)dt = x$$

solution: Applying Ab -Sh T to both sides and using convolution theorem and simplify, we have

$$Q[u(x)] = vs^2 + v^3s^4$$

Operating inverse Ab -Sh T on both sides, we have

$$u(x) = Q^{-1}[vs^2 + v^3s^4]$$
$$= x + \frac{x^3}{6}$$

e) We assume that the linear VI-differential equation of second type

$$u'(x) = 2 + \int_0^x u(t)dt \;\; ;$$
$$u(0) = 2$$

solution: Applying Ab -Sh T to both sides and using initial condition, Using convolution theorem and simplify, we have

$$Q[u(x)] = \frac{2s + 2vs^2}{1 - v^2s^2}$$

Operating inverse Ab-Sh T on both sides, we have

$$u(x) = 2Q^{-1}\left[\frac{s}{1-vs}\right]$$
$$= 2\,e^x$$

f) We assume that the linear VI-differential equation of second type

$$u'''(x) = -1 + \int_0^x u(t)dt \;\; ;$$
$$u(0) = u'(0) = 1\,, u''(0) = -1$$

solution: Applying Ab -Sh T to both sides and using initial condition, Using convolution theorem and simplify, we have

$$Q[u(x)] = \frac{s}{1+v^2s^2} + \frac{vs^2}{1+v^2s^2}$$

Operating inverse Ab -Sh T on both sides, we have

$$u(x) \;= Q^{-1}\left[\frac{s}{1+v^2s^2} + \frac{vs^2}{1+v^2s^2}\right]$$

$$= \; cosx + \sin x$$

Application of Ab –Sh T in Medical Science by linear VIE

During an intravenous injection (continuous) for determining blood glucose concentration $C(x)$ of patient at any particular time t. this concentration $C(x)$ is determined by the following Volterra linear integral equation

$$C(x) = C_i + \left(\frac{\alpha}{V}\right)x - k\int_0^x C(t)\ dt$$

Where

k : constant velocity of elimination , V: volume in which glucose is distributed

α : the rate of infusion , C_i : initial concentration of glucose in the blood

Solution :Applying Ab -Sh T to both sides and using convolution theorem and simplify, we have

$$Q(\,C(x)) = \frac{C_i s + \left(\frac{\alpha}{V}\right)vs^2}{1+kvs}$$

Operating inverse Ab -Sh T on both sides, we have

$$C(x) = C_i e^{-kx} + \frac{\alpha}{kV}\,(1-e^{-kx})$$

Conclusion

In this paper, Using the concept of improper integration, the Abaoub -Shkheam transform was introduced to solve types of Volterra integral equations, as well as to solve the Volterra integral-differential equation of the second type. We review in this research Some of our findings:

- Us Abaoub -Shkheam Transform of Bessel's Functions.
- solve the second type of Volterra equations by Abaoub -Shkheam transform.
- Integral equations are easily solved using Abaoub -Shkheam transform.

References

[1] K. E. Atkinson, "A survey of numerical methods for solving nonlinear integral equations". Integral Equations Appl., 1992, 4, 15–40.

[2] Aggarwal S, Chauhan R, Sharma N. ,"A new application of Kamal transform for solving linear Volterra integral equations", International Journal of Latest Technology in Engineering, Management & Applied Science 2018; 7(4): 138-140.

[3] Aggarwal S, Chauhan R, Sharma N." Application of Elzaki transform for solving linear Volterra integral equations of first kind", International Journal of Research in Advent Technology 2018; 6(12): 3687-3692.

[4] Aggarwal S, Gupta AR. ,"Solution of linear Volterra integro-differential equations of second kind using Kamal transform", Journal of Emerging Technologies and Innovative Research 2019; 6(1): 741-747.

[5] Aggarwal S, Gupta AR, "Sharma SD. Application of Sadik transform for handling linear Volterra integro-differential equations of second kind", *Universal Review* 2019; 10(7): 177-187.

[6] G. Adomian,"Solution of physical problems by decomposition", Appl. Math. Comput. 27(9/10) (1994)145-154.

[7] Ali E. Abaoub, Abejela S. Shkheam, The New Integral Transform "Abaoub-Shkheam transform", Iaetsd Journal For Advanced Research In Applied Sciences ,Volume VII, Issue VI, June/2020, ISSN NO: 2394-8442.

[8] Asmaa Mubayrash , Suad Zali, zaineb alkawash ,"The Use of Abaoub- Shkheam Transform for Solving Partial Differential Equations", 67-74, ALQIRTAS JURNAL, The Twenty   Issue-October 2022.

[9] Suad Zali, ''Application of Abaoub- Shkheam Transform for Solving Linear Partial Integro – Differential Equations", Arab Journal of Sciences & Research Publishing (AJSRP), Vol 9, Issue 1 (2023), P: 108 - 101
ISSN: 2518-5780 (Online) , ISSN: 2518-5780 (Print)

[10] Ali Elhrary Abaoub,"The Adomian Decomposition Method of Volterra Integral Equation of Second Kind", American Journal of Applied Mathematics, 2018; 6(4): 141-147

[11] Abejela S. Shkheam, Ali E. Abaoub, and Yousuf A. Huwaydi, "Exact Solution of Linear Volterra integro-differential Equation of First Kind Using Abaoub-Shkheam Transform", International Journal of Research and Innovation in Applied Science (IJRIAS) |Volume VI, Issue XI, November 2 021|ISSN 2454-6194.

[12] A. Abaoub, and A. Shkheam, "Utilization Abaoub-Shkheam Transform in Solving Linear Integral Equations Of Volterra " journals: International Journal of Software & Hardware Research in Engineering (IJSHRE) ISSN-2347-4890 Volume 8 Issue 12 December 2020.
[13] Lokenath Debnath and Bhatta, D. (2007)" Integral transforms and their applications", Second edition, Chapman & Hall/ CRC.
[14] Aggarwal S, Chaudhary R. A comparative" study of Mohand and Laplace transforms". Journal of Emerging Technologies and Innovative Research 2019; 6(2): 230- 240.
[15] A. M. Wazwaz, " Linear and Nonlinear Integral Equations: Methods and Applications "(Higher Education Press, Beijing, 2011). https://doi.org/10.1007/978-3-642-21449-3
[16] Wazwaz, A.M. "A first course in integral equations", World scientific, Singapore , (1997).