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A Study on the Solution of the Cauchy-Euler Equations

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Abstract:
In this work, we studied the solution of non-homogeneous Cauchy-Euler equations. To find the general solution, we applied the differential transform method to the reduced associated homogenous equation, and to find a particular solution, we applied the differential transform method to non-homogeneous equation. This study showed that this technique is effective for solving the Cauchy-Euler equations and reduced the size of the calculations compared with the usual methods, especially for the higher-order Cauchy-Euler equations.

Keywords: Ordinary differential equations, Cauchy-Euler equations, deferential transform method.

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دراسة في حل معادلات كوشي-أويلر

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الملخص
في هذا العمل قمنا بدراسة حل معادلات كوشي-أويلر الغير متجانسة. لإيجاد الحل العام، طبقنا طريقة التحويل التفاضلي على معادلة كوشي-أويلر المتجانسة المختزلة ولإيجاد الحل الخاص طبقنا طريقة التحويل التفاضلي على معادلة كوشي-أويلر. أظهرت هذه الدراسة أن هذه التقنية فعالة في حل معادلة كوشي-أويلر وقللت من حجم العمليات الحسابية مقارنة بالطرق المعتادة، وخاصة لمعادلات كوشي-أويلر ذات الرتب العليا.

الكلمات المفتاحية: المعادلات التفاضلية العادية، معادلات كوشي-أويلر، طريقة التحويل التفاضلي.

1. Introduction

The differential equations are important to engineers, physicists, mathematicians and researchers, because many physical phenomena and engineering problems are modeled by ordinary or partial differential equations [1-5]. The Cauchy-Euler equations (CEEs) are one of the important types of linear ordinary differential equations [1]. Ref. [6] used the Laplace method applied to find the general solution of the homogeneous CEE [6]. Usually, the variation of parameters method applied to find a particular solution of special types of non-homogeneous CEEs [1]. For CEEs with special bulge function, an expression of the general solution by [7]. Our main aim in this work is to use the differential transform method (DTM) [8-9], to solve the CEEs. In ref. [10], they applied the

DTM to find the particular solutions for some types of Cauchy-Euler ordinary differential equations. And a reliable algorithm for solving CEEs is introduced [11]. The Cauchy-Euler equation has the form

$$\sum_{i=0}^{n} a_i x^i y^{(i)}(x) = g(x) \ . \tag{1}$$

In (1) $a_0, a_1, a_3, ..., a_n$ are constants and we assume that $g(x)$ is an analytic function. The associated homogeneous equation of (1) is

$$\sum_{i=0}^{n} a_i x^i y^{(i)}(x) = 0 \tag{2}$$

To reduced (1) to an equation of constant coefficients, we use the transform

$$x = e^t \tag{3}$$

Hence, the transform (3) enables us to write the derivatives y' , y'' , y''' ,…. in the form

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x}\frac{dy}{dt} \\ \frac{d^2y}{dx^2} &= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) \\ \frac{d^3y}{dx^3} &= \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right), \\ &\vdots \end{aligned} \tag{4}$$

As an example, for $n = 2$, the Cauchy-Euler equation (1) reads

$$a_2 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = g(x) \ , \tag{5}$$

Then by using (4), we can reduce (5) to an equation of constant coefficients of the form

$$a_2 \frac{d^2y}{dt^2} + (a_1 - a_2)\frac{dy}{dt} + a_0 y = f(t), \quad f(t) = g(e^x) \ , \tag{6}$$

In this work, our technique to find the general solution of the CEE (1) is as following: First reduce the associated homogeneous CEE to an equation with constant coefficients, and then we apply the DTM to find the complementary solution. To find a particular solution, we apply the DTM to the CEEs (1).

2. Differential Transform Method

For any analytic function $f(x)$, about $x_0 = 0$, the one-dimensional order differential transform (DT) of $f(x)$ is defined as

$$D_T\{f(x)\} := F(K) = \frac{f^{(k)}(0)}{k!}, \quad k=0,1,2,.. \tag{7}$$

The inverse of $F(k)$ denotes by $D_T^{-1}\{F(k)\}$ and leads to

$$D_T^{-1}\{F(k)\} := f(x) = \sum_{k=0}^{\infty} F(k)x^k \tag{8}$$

Hence, we can prove that the DT (7) satisfies the linearity property

Theorem: $D_T\{\alpha f(x) \pm \beta g(x)\} = \alpha D_T\{f(x)\} \pm \beta D_T\{g(x)\}$, (9)

for any analytic functions $f(x)$ and $f(x)$, and constants α and β. And by following [12], we can list the following fundamental theorems which we need in this work

Theorem: $D_T\left\{\frac{d^n f(x)}{dx^n}\right\} = \frac{(k+n)!}{k!}F(k+n)$ (10)

Theorem: $D_T\{x^m\} = \delta(k,m)$ (11)

Theorem: $D_T\{x^n f^{(n)}(x)\} = \frac{k!}{(k-n)!}F(k)$ (12)

where, $F(k)$ and $G(k)$ are the differential transforms of the functions $f(x)$ and $g(x)$, respectively.

Theorem: $D_T\{f_1(x)\cdot f_2(x)\} = \sum_{n=0}^{k} F_1(n)F_2(k-n)$, (13)

where, $F_1(k)$ and $F_2(k)$ are the differential transforms of $f_1(x)$ and $f_2(x)$, respectively.

3. Numerical Examples

To explain our approach of solving the non-homogeneous Cauchy-Euler equations (1), we study the following examples and compare our solution with solution found by other analytical methods and techniques.

Example (1): Consider the Cauchy-Euler equation

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = 2x^2, \tag{14}$$

The associated Cauchy-Euler equation of (14) reads

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0, \tag{15}$$

To find the solution of (15), we apply the DT on the equation as

$$D_T\left\{\frac{d^2y}{dt^2}\right\} - 2D_T\left\{\frac{dy}{dt}\right\} - 3D_T\{y\} = 0 \tag{16}$$

Then by using (9) and (10), we can write

$$(k+1)(k+2)Y(k+2)-2(k+1)Y(k+1)-3Y(k)=0. \tag{17}$$

Next for (17), we can write the iteration formula:

$$Y(k+2)=\frac{2}{(k+2)}Y(k+1)+\frac{3}{(k+2)(k+1)}Y(k), \qquad k=0,1,2,..... \tag{18}$$

Now by using (18), we can calculate

$$Y(2)=Y(1)+\frac{3}{2}Y(0), \quad Y(3)=\frac{7}{6}Y(1)+Y(0), \quad Y(4)=\frac{5}{6}Y(1)+\frac{7}{8}Y(0), \quad \cdots \tag{19}$$

This gives the complementary solution as

$$y_c(t)=Y(0)\left(1+\frac{3}{2}t^2+t^3+\frac{7}{8}t^4+...\right)+Y(1)\left(t+t^2+\frac{7}{6}t^3+\frac{5}{6}t^4+...\right) \tag{20}$$

To simplify the solution (20), we set

$$Y(0)=c_1+c_2, \qquad Y(1)=-c_1+3c_2.$$

Where c_1 and c_2 are constants. This gives

$$y_c(t)=c_1\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-...\right)+c_2\left(1+3t+\frac{9}{2!}t^2+\frac{27}{3!}t^3+\frac{81}{4!}t^4+...\right). \tag{21}$$

Then, we can conclude the close form complementary solution of (15) is

$$y_c(t)=c_1e^{-t}+c_2e^{3t}, \tag{22}$$

Therefore, the close form complementary solution of associated homogeneous equation of (14) can be writing in the form

$$y_c(x)=c_1x^{-1}+c_2x^3, \tag{23}$$

Next, to find a particular solution, we apply the DT on both sides of the CEE (14) and using theorems (9-12). This gives the iteration formula

$$Y(k)=\frac{2\delta_{k2}}{k(k-1)-k-3}, \quad k=0,1,2,..., \tag{24}$$

which yields $Y(2)=-2/3$ for $k=2$ and $Y(k)=0$ for $k\neq 2$. This leads to the solution particular solution

$$y_p(x)=-\frac{2}{3}x^2 \tag{25}$$

At the end (23) and (25) lead to the general solution

$$y(x)=c_1x^{-1}+c_2x^3-\frac{2}{3}x^2 \tag{26}$$

Note that the auxiliary equation of (15) leads to the complementary solution (22) and hence (23) and any usually method, will lead to the particular solution (25), [1].

Example (2): Consider the second order equation

$$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = 2x^4e^x, \tag{27}$$

The solution of the CEE (27) can be found by using the usual analytic methods. In ref. [1], the general solution is found in the form

$$y(x) = \tilde{c}_1 x + \tilde{c}_2 x^3 + 2x^2e^x - 2xe^x, \tag{28}$$

which can be simplify as following:

$$\begin{aligned} y(x) &= \tilde{c}_1 x + \tilde{c}_2 x^3 + 2\sum_{i=0}^{\infty} \frac{1}{i!}x^{i+2} - 2\sum_{i=0}^{\infty} \frac{1}{i!}x^{i+1} \\ &= \tilde{c}_1 x + \tilde{c}_2 x^3 + x^3 - 2x + \left(\frac{4}{3!}x^4 + \frac{6}{4!}x^5 + \frac{8}{5!}x^6 + \frac{10}{6!}x^7 \ldots \right) \end{aligned} \tag{29}$$

Or if we set $c_1 = (\tilde{c}_1 - 2)$ and $c_2 = (\tilde{c}_2 + 1)$ reads

$$y(x) = c_1 x + c_2 x^3 + \left(\frac{4}{3!}x^4 + \frac{6}{4!}x^5 + \frac{8}{5!}x^6 + \frac{10}{6!}x^7 \ldots \right). \tag{30}$$

Where c_1 and c_2 are constants.

To apply our approach, we use (4), to reduce the associated CEE (27) to the form

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 0, \tag{31}$$

Then, we apply the DT as

$$D_T\left\{\frac{d^2y}{dt^2}\right\} - 4D_T\left\{\frac{dy}{dt}\right\} + 3D_T\{y\} = 0 \tag{32}$$

Next, we use (9) and (10), to write can write the iteration formula

$$Y(k+2) = \frac{4}{(k+2)}Y(k+1) - \frac{3}{(k+2)(k+1)}Y(k), \qquad k = 0,1,2,3,\ldots \tag{33}$$

Now, from the iteration formula (33) we can calculate

$$\begin{aligned} Y(2) &= 2Y(1) - \frac{3}{2}Y(0), \\ Y(3) &= \frac{13}{6}Y(1) - 2Y(0), \end{aligned} \tag{34}$$

$$Y(4)=\frac{5}{3}Y(1)-\frac{13}{8}Y(0),$$
$$\vdots$$

This leads to the complementary solution as following:

$$y_c(t)=Y(0)+Y(1)t+2\,Y(1)t^2-\frac{3}{2}Y(0)t^2+\frac{13}{6}Y(1)t^3-2\,Y(0)t^3+\frac{5}{3}Y(1)t^4-\frac{13}{8}Y(0)t^4+\dots$$

or

$$y_c(t)=Y(0)\left(1-\frac{3}{2}t^2-2t^3-\frac{13}{8}t^4+\dots\right)+Y(1)\left(t+2t^2+\frac{13}{6}t^3+\frac{5}{3}t^4+\dots\right) \quad (35)$$

Hence, if we set $Y(0)=k_1+k_2$ and $Y(1)=k_1+3k_2$ in (35), then we can write

$$y_c(t)=k_1\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\dots\right)+k_2\left(1+3t+\frac{9}{2!}t^2+\frac{27}{3!}t^3+\frac{81}{4!}t^4+\dots\right) \quad (36)$$

Where k_1 and k_2 are constants. This leads to the close form solution

$$y_c(t)=k_1e^t+k_2e^{3t} \quad (37)$$

And hence the close form complementary solution of (27) is

$$y_c(x)=k_1x+k_2x^3 \quad (38)$$

To find a particular solution, we apply the DT on the both sides of the CEE (27). Then by using (9-10) and theorem (13), we can write

$$k(k-1)Y(k)-3kY(k)+3Y(k)=(k-1)(k-3)Y(k)=2\sum_{n=0}^{k}\delta_{n4}\frac{1}{(k-n)!}. \quad (39)$$

By using (39) we can conclude that $Y(k)=0$ when $k=0,1,2,3$, and for $k=4,5,6,\dots$ gives the iteration formula

$$Y(k)=\frac{2}{(k-1)(k-3)}\left[\frac{1}{(k-4)!}\right] \quad (40)$$

Next, a particular solution of (27) can be written in the form

$$y_p(x)=\frac{2}{3}\frac{1}{0!}x^4+\frac{2}{8}\frac{1}{1!}x^5+\frac{2}{15}\frac{1}{2!}x^6+\frac{2}{24}\frac{1}{3!}x^7+\dots.. \quad (41)$$

At the end the (38) and (41) give the general solution

$$y(x)=k_1x+k_2x^3+\left(\frac{2}{3}\frac{1}{0!}x^4+\frac{2}{8}\frac{1}{1!}x^5+\frac{2}{15}\frac{1}{2!}x^6+\frac{2}{24}\frac{1}{3!}x^7+\dots..\right), \quad (42)$$

which is identical to general solution (29) found by the usual methods.

Example (3): Consider the CEE

$$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 13y = 4 + 3x\,, \tag{43}$$

First, the reduced associated homogeneous equation of the CEE (43) reads

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 0\,, \tag{44}$$

To find the solution, we apply DT on both sides of (44). This implies

$$D_T\left\{\frac{d^2y}{dt^2}\right\} - 4D_T\left\{\frac{dy}{dt}\right\} + 13D_T\left\{y\right\} = 0 \tag{45}$$

Then by using (9-10), we can find iteration formula

$$Y\left(k+2\right) = \frac{4}{\left(k+2\right)}Y\left(k+1\right) - \frac{13}{\left(k+2\right)\left(k+1\right)}Y\left(k\right), \qquad k = 0,1,2,..... \tag{46}$$

Next using (46) for $k = 0,1,2,....$, we can calculate

$$Y(2) = 2Y\left(1\right) - \frac{13}{2}Y\left(0\right), \quad Y(3) = \frac{1}{2}Y\left(1\right) - \frac{26}{3}Y\left(0\right), \quad Y(4) = -\frac{5}{3}Y\left(1\right) - \frac{13}{8}Y\left(0\right), \quad \cdots \tag{47}$$

Hence, the close form complementary solution of (44) can be written in the form

$$y_c(t) = Y\left(0\right) + Y\left(1\right)t + 2Y\left(1\right)t^2 - \frac{13}{2}Y\left(0\right)t^2 + \frac{1}{2}Y\left(1\right)t^3 - \frac{26}{3}Y\left(0\right)t^3 - \frac{5}{3}Y\left(1\right)t^4 - \frac{13}{8}Y\left(0\right)t^4 + ... \tag{48}$$

or

$$y_c(t) = Y\left(0\right)\left(1 - \frac{13}{2}t^2 - \frac{26}{3}t^3 - \frac{13}{8}t^4 + ...\right) + Y\left(1\right)\left(t + 2t^2 + \frac{1}{2}t^3 - \frac{5}{3}t^4 + ...\right) \tag{49}$$

For simplicity, we set $Y\left(0\right) = c_1$ and $Y\left(1\right) = 2c_1 + 3c_2$ in (49). This gives

$$y_c(t) = c_1\left(1 + 2t - \frac{5}{2}t^2 - \frac{23}{3}t^3 - \frac{119}{24}t^4 + ...\right) + c_2\left(3t + 6t^2 + \frac{3}{2}t^3 - 5t^4 + ...\right) \quad , \tag{50}$$

Which is the Taylor series expansion of

$$y_c\left(t\right) = c_1e^{2t}\cos 3t + c_2e^{2t}\sin 3t\ , \tag{51}$$

Therefore, the close form complementary solution of associated homogeneous equation of (43) can be writing in the form

$$y_c\left(x\right) = x^2\left[c_1\cos\left(3\ln x\right) + c_2\sin\left(3\ln x\right)\right], \tag{52}$$

For the particular solution, we apply the DT on both sides of the CEE (43) and using (9-12). This gives the iteration formula

$$Y(k)=\frac{4\delta_{k0}+3\delta_{k1}}{k(k-1)-3k+13},\quad k=0,1,2,.....\tag{53}$$

Using (53), conclude that $Y(k)=0$ for $k=2,3,4,...$ and for $k=0,1$, we can find that $Y(0)=4/13$ and $Y(1)=3/10$. This leads to the particular solution

$$y_p(x)=\frac{4}{13}+\frac{3}{10}x\tag{54}$$

Which with (54) lead to the general solution of the CEE (43). Note that the auxiliary equation of the reduced equation (44) leads to (51) and hence (52). Furthermore, and any analytical method, such as the variation of parameters method or the undetermined coefficients method, will lead to the particular solution (54).

Conclusion

In this work, the DTM has been successfully applied to find the solution of the CEEs. The numerical examples showed that our technique is powerful for solving CEEs and reduced the size of the calculations comparing with the usually methods. This study also showed that this technique will be useful and effective for solving the higher order Cauchy-Euler equations.

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