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Axioms of R-Countability Via R-Dense Sets

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مسلمات العد المنتظمة عبر مجموعات الكثيفة المنظمة

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Abstract:
In this article, we introduce a new type of r-countability axiom by using the concept of regular-dense (r-dense) sets; the so called D-r-countability axiom, which states that space is D-r-(separable, first countable, Lindelöf, δ-compact, second countable) if a topological space contains an r-(separable, first countable, Lindelöf, δ-compact, second countable) subspace that is r-dense. We study the relationship between these spaces, the r-countability axiom and the d-countability axiom. Furthermore, we characterize the topological hereditary properties of these spaces using theory and some examples. Finally, we study the behavior of the D-r-countability axiom and D-countability axiom in regular spaces.

Keywords: Dense set, Separable space, first countable space, δ-compact space, Lindelöf space.

الملخص
نقدم نوعًا جديدًا من مسلمات العد المنتظمة باستخدام مفهوم المجموعات الكثيفة المنتظمة؛ تسمى مسلمات العد الكثيفة المنتظمة، تنص هذه المسلمات على أن الفضاء يعتبر قابل للفصل كثيف منتظم (مسلمة العد الأول كثيف منتظم، ليندلوف كثيف منتظم، δ-متراص كثيف منتظم، مسلمة للعد الثانية كثيف منتظم) إذا كان الفضاء الطوبولوجي يحتوي على فضاء جزئي قابل للفصل منتظم (مسلمة للعد الأول منتظم، ليندلوف منتظم، δ-المتراص منتظم، مسلمة العد الثانية منتظم) الذي هو كثيفة منتظم.
قمنا بدراسة العلاقة بين هذه الفضاءات، مسلمات العد الكثيفة ومسلمات العد المنتظمة. بالإضافة إلى الخصائص الوراثية الطوبولوجية لهذه الفضاءات اعتمادا على النظريات وبعض الأمثلة. بالإضافة إلى ذلك، فإننا ندرس سلوك مسلمات العد الكثيفة المنتظمة ومسلمات العد الكثيفة في الفضاء المنتظم، ملقين الضوء على التعقيدات والجوانب المثيرة للاهتمام في هذا المجال.

الكلمات المفتاحية: المجموعات الكثيفة، فضاء قابل الفصل، مسلمة العد الاولي، فضاء δ- المتراصة، فضاء ليندلوف.

Introduction

Many investigations have been conducted to define new r-countability axiom, some of which are weaker than others. The important classical properties of topological spaces include r-separable, r-first countable, r-Lindelöf, r-δ-compact and r-second countable space. These properties are described in detail in [1]. In 1974, Siwiec [3] defined g-first countable and second countable spaces in 1974 by using the notion of a susceptible base in topological space. He then examined the relationship between these concepts and metrizability. A year last, Siwiec [4] examined the theories that broaden the notion of first countability. He also looked at the relationship between these theories, stating that a space is first countable if it is both Frechet g-first countable and [4]. Through b-open sets [6], Selvarani [5] presented the b-countability axioms principles in 2013. In same year, a set of axioms related to countability on GT—known as the μ-countability axioms—was developed by Ayawan and Canoy [7]. The μ-first and μ-second properties of the product of GT's are the properties that are connected with these concepts when viewed as concepts. Further details about universal topological bases' characteristics can be found in [8, 9]. Pre-countability principles are a subset of μ-countability principles that were defined in 2021 by Elbhilil and Arwini [10] using the pre-open notion. They also examined the topological characteristics of these spaces. The term "s-countability axioms" refers to the overall framework for measuring countability, that Elbhilil [11] defined in 2023 using the idea of semi-open sets. [2, 1], Arwini and Kornas introduced two types of generalizations of countability axioms. The primary type, known as D-countability axioms, is defined using idea of dense sets. They demonstrated that D-separable, separable, and D-second countable spaces are equal, and they provided several examples illustrating the relationships between D-countability axioms and traditional countability axioms. The second class, termed R-countability axioms, utilizes the notion of regular open sets for its definition. They explored the properties of R-countability axioms and proved that R-countability axioms and traditional countability axioms coincide in regular spaces [12, 13 and 14].

In this newsletter, we introduce a new type of r-countability axiom through using idea of regular dense (r-dense) units; the so referred to as D-r-countability axiom, which states that area is D-r-separable, D-r-first countable space, D-r-Lindelöf, D-r-δ-compact, and D-r-second countable space. We explore the relationships among these D-r-countability axioms and their connections to classical r-countability axioms as well as D-countability axioms. Additionally, we characterize the topological hereditary properties of these spaces and investigate their behavior in certain special contexts, such as regular and locally compact spaces.

2. D-R-SEPARABLE SPACES

Definition 2.1. [1] If the topological space Z contains a countable r-dense subset, then Z is called r-separable.

Definition 2.2. [2]] If the topological space Z contains a dense separable subspace, it is called D-separable.

Definition 2.3. If the topological space Z contains an r-dense r-separable subspace, it is called D-r-separable.

Corollary 2.1. [2] If Z a topological space, then separable space and D-separable are equivalent

Theorem2.1. If Z topological space, then these conditions are equivalently:
1. Z is r-separable.
2. Z is D-r-separable.

Proof
$\Longrightarrow$ Obvious.
$\Longleftarrow$ According to definition 2.3, N is r-dense r-separable subspace, meaning N contains a countable r-dense subset M, and thus M is r-dense subset in Z. There fore Z is r-separable.

3. D-R-FIRST COUNTABLE SPACES

Definition 3.1. [1] A topological space Z is said to be r-first countable space if for every z ∈ Z there is a countable r-local base $\mathfrak{B}_z$ at z.

Definition 3.2. [2] If a topological space Z has first countable subspace that is dense, then Z is called D-first countable.

Definition 3.3. If a topological space Z has an r-first countable subspace that is r-dense, then Z is called D-r-first countable.

Examples 3.1.

1. If Z= ℝ, and τ= {U⊆ ℝ: U^C is countable}, then (ℝ, τ) is r-first countable and D-r-first countable but not D-first countable.
2. If Z= ℝ, and τ= discrete topology, then (ℝ, τ) is D-r-first countable but not r-separable.
3. The topological space $[0,w_1]$ is D-first countable and D-r-first countable, but not r-first countable nor r-separable

Propositions 3.1.

1. Any r-first countable space is D-r-first countable.
2. Any D-first countable space is D-r-first countable.

Proof:
1. Obvious.
2. Obvious, since any first countable space is r-first countable.

Theorem 3.1. Any D-r-first countable in regular space is D-first countable.

Proof: Obvious, since Z is regular space.

.D-r-first countable $\xrightarrow{\text{regular}}$ D-first countable

We summarize the relations between D-r-first countable, D-first countable and r-first countable in diagram 1.

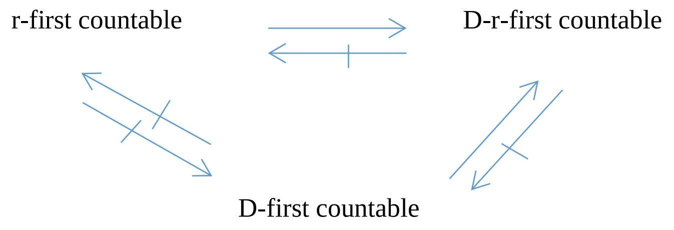

Diagram 1. Relations between D-r-first countable, D-first countable and r-first countable.

Theorem 3.2. Let Z be a D-r-first countable space and N be an r-open subset of Z, then N is D-r-first countable subspace in Z.

Proof: Suppose that N ⊆ Z, such that N is an r-open subspace of Z, then Z has an r-first countable and r-dense subspace M. First: we will show that N∩M is D-r- first countable, i.e. N∩M ≠ ϕ and N∩M ⊂ M, then N∩M is r- first countable. Now we will show that N∩M is r-dense. Let U be a nonempty r-open subset in N, and N is an r-open subset Z, then U is an r-open subset Z, which implies that U∩M ≠ ϕ and U∩(N∩M) ≠ ϕ, therefore, N∩M is r-dense, hence N is D-r- first countable.

Theorem 3.3. If N D-r-first countable subspace of Z, that is r-dence, then Z is D-r-first countable space.

Proof: If N is a D-r-first countable and r-dense subspace of Z, by definition 3.3. N contains a r-first countable r-dense subspace V, which means that V is r-first countable and r-dense in Z. Therefore Z is D-r-first countable.

Corollary 3.1. Any r-separable space is D-r-first countable.

r-separable ⟶ D-r-first countable

4. D-R-LINDELÖF SPACES

Definition 4.1. [1] A topological space Z is called r-Lindelöf (or nearly Lindelöf space) if for every r-open cover of Z, there exists a countable subcover.

Definition 4.2. [2] If the topological space Z contains a dense Lindelöf subspace, then Z is called D- Lindelöf.

Definition 4.3. If the topological space Z contains an r-dense r-Lindelöf subspace, then Z is called D-r-Lindelöf.

Examples 4.1.

1. The topological space $[0,w_1]$ is D-r- Lindelöf and r- Lindelöf but not r-separable.
2. The topological space $[0,w_1)$ is D-r-Lindelöf and r-Lindelöf but not D-Lindelöf.
3. The sorgenfery plane $\mathbb{R}_l \times \mathbb{R}_l$ is D-r-Lindelöf and D-Lindelöf but not r-Lindelöf.
4. Let Z = ℝ, and τ={Z}∪{V⊆ℝ: 1∉V}, then Z is not D-r- Lindelöf.

Proposition 4.1.

1. Any r-Lindelöf space is D-r-Lindelöf.
2. Any D-Lindelöf space is D-r-Lindelöf.

.**Proof:**
1. Obvious.
2. Obvious, since any Lindelöf space is r-Lindelöf.

Theorem 4.1. D-r-Lindelöf in regular space is D- Lindelöf.

Proof: Obvious, since Z is regular space.

D-r-Lindelöf $\xrightarrow{\text{regular}}$ D-Lindelöf

We summarize the relations between D-r-Lindelöf, D-Lindelöf and r-Lindelöf in diagram 2.

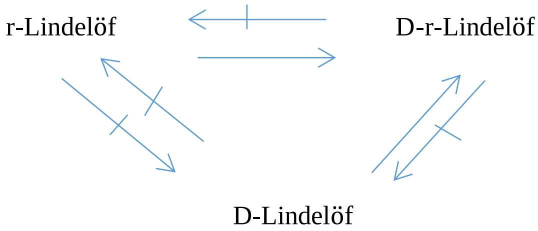

Diagram 2. Relations between D-r-Lindelöf, D-Lindelöf and r-Lindelöf.

Theorem 4.2. Let Z be a D-r-Lindelöf space and N be an r-clopen subset of Z, then N is D-r-Lindelöf subspace of Z.

Proof: : Assume N ⊆ Z, such that N is an r-clopen subspace of Z, then Z has an r-δ- Lindelöf and r-dense subspace M, First: we will show that N∩M is D-r- Lindelöf, means that N∩M ≠ ϕ and N∩M ⊂ M, thus N∩M is r-δ- Lindelöf. Now we will show that N∩M is r-dense. Let U be a nonempty r-open subset of N, and since N be an r-open subset of Z, U is also an r-open subset of Z, which implies that U∩M ≠ ϕ and U∩(N∩M) ≠ ϕ, therefore N∩M is r-dense, so N is D-r-δ- Lindelöf.

Theorem 4.3. If N D-r- Lindelöf subspace of Z, which is r-dence, then Z is a D-r-Lindelöf space.

Proof: If N is a D-r-Lindelöf and r-dense subspace of Z, by definition 4.3. N contains a subspace that is r-Lindelöf and r-dense, thus V is an r-Lindelöf and is r-dense in Z. Therefore Z is D-r- Lindelöf.

Corollary 4.1. Any r-separable space is D-r-Lindelöf.

r-separable $\longrightarrow$ D-r-Lindelöf

5. D-R-δ-COMPACT SPACES

Definition 5.1. [1] A topological space Z is called r-δ-compact if it is the union of a countable many r-compact subset of Z.

Definition 5.2. [1] If the topological space Z contains a dense δ-compact subspace, then Z is called D- δ-compact.

Definition 5.3 If the topological space Z contains an r-dense r-δ-compact subspace, then Z is called D-r-δ-compact.

Examples 5.1.

1. The topological space $[0,w_1]$ is D-r-δ-compact but not r-separable.
2. The sorgenfery line $\mathbb{R}_l$ is D-r-δ-compact and D-δ-compact but not r-δ-compact.
3. If Z = ℝ, and τ= {U⊆ ℝ: U^C is countable}, then (ℝ, τ) is D-r-δ-compact and r-δ-compact but not D-δ-compact.
4. If Z = ℝ, and τ= discrete topology, then (ℝ, τ) is not D-r-δ-compact.

Proposition 5.1.

1. Any r-δ-compact space is D-r-δ-compact.
2. Any D-δ-compact space is D-r-δ-compact.

Proof:

1. Obvious.
2. Obvious, since any δ-compact space is r-δ-compact.

Theorem 5.1. D-r-δ-compact in regular space is D-δ-compact.

Proof: Obvious, since Z is regular space.

$$\text{D-r-}\delta\text{-compact} \xrightarrow{\text{regular}} \text{D-}\delta\text{-compact}$$

We summarize the relations between D-r-δ-compact, D-δ-compact and r-δ-compact in diagram 3.

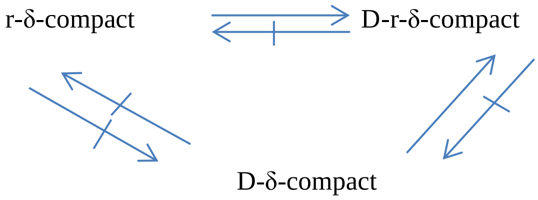

Diagram 3. Relations between D-r-δ-compact, D-δ-compact and r-δ-compact.

Theorem 5.2. Let Z be a D-r-δ-compact space and N be an r-clopen subset of Z, then N is D-r-δ-compact subspace in Z.

Proof: Assume N ⊆ Z, such that N is an r-clopen subspace of Z, then Z has an r-δ-compact and r-dense subspace M, First: we will show that N∩M is D-r-δ-compact, i.e. N∩M ≠ ϕ and N∩M ⊂ M, then N∩M is r-δ-compact. Now we will show that N∩M is r-dense. Let U be a nonempty r-open subset of N, and N be an r-open subset of Z, then U is an r-open subset of Z, i.e. U∩M ≠ ϕ and U∩(N∩M) ≠ ϕ, then N∩M is r-dense, so N is D-r-δ-compact.

Theorem 5.3. If N D-r-δ-compact subspace of Z, that's r-dence, then Z is D-r-δ-compact space.

Proof: If N is D-r-δ-compact and r-dense subspace of Z, by definition 5.3. N has r-δ-compact and r-dense subspace V, then V is r-δ-compact which is r-dense in Z. Thus Z is δ-compact.

Corollary 5.2.

1. Any r-separable space is D-r-δ-compact.
2. Any D-r-δ-compact space is D-r- Lindelöf.

$$\text{r-separable} \longrightarrow \text{D-r-}\delta\text{-compact}$$

$$\text{D-r-}\delta\text{-compact} \longrightarrow \text{D-r-Lindelöf}$$

Theorem 5.4. D-r-Lindelöf in regular locally compact T_2 space is D-δ-compact.

Proof: From the definition of D-r- Lindelöf space, N is r-dense r-Lindelöf subspace of Z, since Z is regular (locally compact T_2) space, then N is dense (δ-compact) subset in Z, Thus Z is D-δ-compact.

$$\text{D-r- Lindelöf} \xrightarrow{\text{regular locally compact } T_2} \text{D-}\delta\text{-compact}$$

Corollary 5.3. D-r-Lindelöf in locally compact T_2 space is D-r-δ-compact.

$$\text{D-r- Lindelöf} \xrightarrow{\text{locally compact } T_2} \text{D-r-}\delta\text{-compact}$$

6. D-R-SECOND COUNTABLE SPACES

Definition 6.1. [1] If topological space Z has a countable base, then Z is called r-second countable.

Definition 6.2. [2] If the topological space Z contains dense second countable subspace, then Z is called D-second countable.

Definition 6.3. If the topological space Z contains r-dense r-second countable subspace, then Z is called D-r-second countable.

Corollary 6.1. Any D-r-second countable space is D-r-first countable (D-r- Lindelöf).

Example 6.1. In topological space $[0,w_1]$ is D-r-first countable and D-r-Lindelöf but not D-r-second countable.

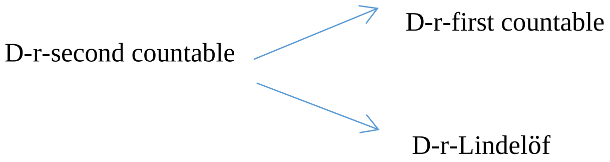

Diagram 4. Relations between D-r-second countable, D-r-first countabl and D-r- Lindelöf.

Corollary 6.2. If Z topological space, then these condition are equivalently:
1. Z is D-r-second countable space.
2. Z is r-separable space.
3. Z is D-r-separable space.

Proof:
$1 \Longrightarrow 2$) Obvious
$2 \Longrightarrow 3$) Obvious, by theorem 2.1.
$3 \Longrightarrow 1$) By definition 2.3. Z has r-separable and r-dense subspace N, then M is a countable r-dense subset of N, i.e. M is r-second countable and r-dense subset in Z. Hence Z is D-r second countable.

D-second countable ⟷ r-separable ⟷ D-r-separable

Conclusion

In this paper, using the idea of r-dense sets, we have introduced a new concept of r-countability, the so called D-r-countability axiom. We have shown that these spaces are weaker than the D-countability axiom (r-countability axiom). We have shown that the D-countability axiom (r-countability axiom) is D-r-countability, but not vice versa. Furthermore, we have shown that there is no relationship between the D-countability axiom and r-countability. We have shown that r-separable spaces, D-r-separable spaces and D-r-second countable spaces are equivalent. Furthermore, we have shown that the D-r-countability axioms and D-countability axioms are equivalent when the space is regular. We have explained the relationship between these spaces; every separable space is D-r-first countable (D-r- Lindelof D-δ-r-compact), every D-r-δ-compact space is D-r-Lindelöf, and in (regular) locally compact T_2 spaces, D-R-Lindelöf space is D-r-δ-compact (D-δ-compact).
. R-Clopen subspace of D-r- Lindelöf (D-r- δ-compact) is D-r-Lindelöf (D-r-δ-compact).
. An R-Open subspace of D-r-first countable is D-r-first countable.

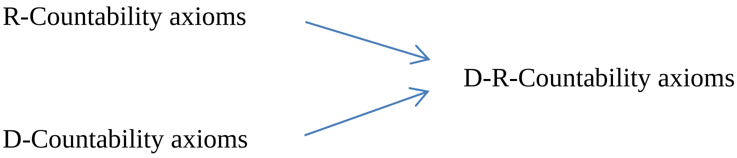

Diagram 5. Relations between D-r-countability axiom, D-countability axiom and r-countability axiom.

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